GENERAL PHYSICS AND SOUND

PHYSICAL SCIENCE TEXTS

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GENERAL PHYSICS AND SOUND

To Advanced and Scholarship Level

By

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THE ENGLISH UNIVERSITIES PRESS LTD

102 NEWGATE STREET

LONDON, E.C. 1

First printed . . 1957

PRINTED AND BOUND IN ENGLAND BY
HAZELL WATSON AND VINEY LTD
AYLESBURY AND LONDON

GENERAL EDITOR'S FOREWORD

General Editor:

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Director, Meteorological Office;
formerly Dean of the Royal Military College of Science, Shrivenham,
and Bashforth Professor of Mathematical Physics.

THE present volume is one of a series on physics and mathematics for the upper forms at school and the first year at the university. The books have been written by a team of experienced teachers at the Royal Military College of Science where, among other things, students are prepared for London (External) Degrees in Natural Science and Engineering. The series therefore forms an integrated course of study based on many years' experience in the teaching of physics and mathematics.

In preparing their manuscripts the writers have been mainly guided by the examination syllabuses of London University, the Joint Board of Oxford and Cambridge and the Northern Universities Board, but they have also taken a broad view of their tasks and have endeavoured to produce works which aim to give a student that solid foundation without which it is impossible to proceed to higher studies. The books are suitable either for class teaching or self study; there are many illustrative examples and large collections of problems for solution, taken, in the main, from recent examination papers.

It is a truism too often forgotten in teaching that knowledge is acquired by a student only when his interest is aroused and maintained. The student must not only be shown how a class of problems in mathematics is solved but, within limits, why a particular method works, and, in physics, why a technique is especially well adapted for some particular measurement. Throughout the series special emphasis has been laid on illustrations which may be expected to appeal to the experience of the student in matters of daily life, so that his studies are related to what he sees, feels and knows of the world around him. Treated in this way, science ceases to be an arid abstraction and becomes vivid and real to the enquiring mind.

The books have therefore been written, not only to ensure the passing of examinations, but as a preparation for the exciting world which lies ahead of the reader. They incorporate many of the suggestions which have been made in recent years by other teachers and, it is

hoped, will bring some new points of view into the classroom and the study. Last, but by no means least, they have been written by a team working together, so that the exchange of ideas has been constant and vigorous. It is to be hoped that the result is a series which is adequate for all examinations at this level and yet broad enough to satisfy the intellectual needs of teachers and students alike.

O. G. SUTTON

PREFACE

THIS book has been written mainly for students preparing for the General Certificate of Education at Advanced or Scholarship level during their Sixth Form years in school, and for students who in their first year at a University are working for an examination at Intermediate level. The work contained in this book is based on the syllabuses issued by the various examining bodies. In particular, the General Certificate syllabuses of the following have been used: University of London, University of Oxford Delegates for Local Examinations, University of Cambridge Local Examinations Syndicate, Oxford and Cambridge Schools Examination Board and the Northern Universities Joint Matriculation Board. In addition, work is included to cover the Entrance Scholarship examinations for the Universities of Oxford, Cambridge and Manchester.

The standard of the questions set by these examining bodies covers a very large range, many being of degree standard. It is very tempting to write up to the standard of these degree questions on the assumption that the basic work has been done previously. However, it must be realised that candidates capable of really advanced work are rare, and in any case deserve individual attention and very careful teaching. The main body of students is not of this standard, indeed many candidates working at this level are studying Physics for the first time. These include the students who do not intend to become physicists but enter a University to read some other science and have first to pass an examination in Physics at Intermediate level. Such candidates often need some fairly advanced ideas in Physics but without the initial groundwork; for example, a student of Botany is helped by an understanding of the processes of osmosis and diffusion but does not wish to be encumbered with the basic ideas of the kinetic theory of matter.

The approach in writing this book has been firstly to sketch in the elementary theory of the subject; sufficient detail is given in each chapter to satisfy the non-specialist studying the subject for the first time, or to act as revision for the Physics student who has already taken the Ordinary Level examinations. Next a careful explanation is given in everyday terms of those physical phenomena which should be understood by both classes of student. Finally, the theoretical treatment of suitable parts of the syllabus is extended to the limits of the good Advanced Level or Scholarship candidate.

Research in Physics is progressing at an ever increasing rate; this has resulted in more and more work being moved into the early syllabus in order to leave time to teach the recent advances of modern Physics

in the final years of a University course. As a consequence, it is felt by many authorities that the Advanced Level and Intermediate examination syllabuses are seriously overloaded. Physics, above all, must be taught systematically; experimental evidence is gathered together, logical deductions are made from the experiments and these deductions are tested by further experiment. In this book only those experiments which are essential to the logical development of the subject are described, many which are now only of an historical interest have been omitted. Economy of this nature has enabled much recent work to be described without overloading the student unduly.

The book is divided into three sections. Chapters 1 to 4 deal with Applied Mathematics or Mechanics, including rotational and simple harmonic motion. Chapters 5 to 9 describe the properties of matter; that is, gravitation, hydrostatics, the properties of gases, elasticity and the properties of fluids, surface tension, diffusion and viscosity. The remaining three chapters are devoted to the study of sound. Chapter 10 describes the properties of travelling waves, Chapter 11 those of standing waves and finally, in Chapter 12, the principles derived in the previous two chapters are applied to various acoustical instruments and systems.

I am indebted to the examining bodies mentioned earlier for permission to reprint examination questions. I should also like to acknowledge with pleasure my thanks to my colleagues for assistance during the writing of this book: to Dr A. J. Woodall and Mr C. B. Daish for their helpful advice and criticism, to Dr Stella Mayne and Mr R. M. Pritchard who read the proofs, and finally to the General Editor of this series, Sir Graham Sutton, F.R.S., for the interest he has taken in the preparation of this book.

D. H. FENDER

CONTENTS

CHAPTER 1

BASIC THEORIES AND QUANTITIES

1.1 Introduction

The earliest members of our civilisation must have run into the problems of measurement—how far to the next village?—how much corn is stored for the winter? and so on. In the course of time assessment of size or quantity has become instinctive in all of us; but the process of measurement is more than just a social expedient, it provides the basis of a scientific method—the science of Physics.

Physics is concerned with all those properties of inanimate matter which can be measured; properties such as length, weight and many more to be described later. We also notice certain phenomena associated with the behaviour of inanimate matter; for example a solid body gets larger if heated or the volume of a gas changes with pressure. Some of these phenomena can be measured and so related to the properties of matter; this is the basic logic of Physics.

To describe these properties or phenomena in the most concise way, the methods and symbolism of mathematics are used to summarise what we observe; as an example, the very simple algebraic equation

$$s = vt \qquad (1)$$

can represent quite a complex physical statement, for if s represents the distance travelled by a body in time t when moving at a steady speed v, the equation is seen to be a shorthand notation for 'the distance travelled by a body in time t when moving at a speed v is equal to the product of time and speed'.

Writing a physical statement as an equation implies that there are certain properties of matter which can be linked with numbers.

One physical statement usually draws together a group of phenomena and occasionally we find that such a statement is true in all circumstances, we then have a *natural law*. As an example, an outline is given here of the reasoning which leads to the law of universal gravitation; a more complete discussion is given in Chapter 5.

The first observed phenomenon is that an unsupported body falls towards the Earth; this naturally leads to the idea that the body is attracted by the Earth and then that *any* two bodies attract each other. The attraction between the two bodies can be measured by a quantity called *Force*.

The hypothesis is then extended to the statement that the amount of matter in the bodies is the only physical property of the bodies

which influences the magnitude of the force; also that the force decreases with increasing distance between the bodies but does not depend in any way on the nature of the material which lies between them. The force would thus exist equally well in empty space.

So far this is purely descriptive and amounts to no more than a literary statement. Newton put the proposition into exact mathematical form; he supposed that the amount of matter in a body could be measured by a quantity called the *mass* of the body, and if two bodies of mass m_1 and m_2 respectively are a distance d apart, then the force between them is given by

$$F = \frac{G\, m_1\, m_2}{d^2} \quad . \quad . \quad . \quad . \quad (2)$$

where G is some constant.

If means are available for measuring force and mass as well as distance, then this statement can be tested by experiment, and the value of G calculated. This has been done and G has been found to have the same value in all tests, however varied their nature.

Thus the original speculative statement has become a law which is universally true, and which links together the physical quantities of force, mass, distance and the gravitational constant G.

If numbers are to be used to represent *physical magnitudes* (such as distance, speed, time, weight, temperature, area, etc.), we must primarily select and name those properties of matter, or phenomena associated with matter, which can easily be described by numbers. The quantities mentioned above come more or less into this category—ones which appear to fall outside it are stickiness, smell, the 'consistency' of a body such as cheese, etc., although as our knowledge increases it becomes more and more possible to describe even quantities such as these by numbers.

Further, we must devise a system whereby all scientists, no matter when or where they work, will attach the same number to the same physical magnitude; this is the process called *measurement* and provides us with a direct method of comparing properties.

If physical properties are to be measured, we must have some properties of *standard size* for comparison. For convenience we should restrict these standards to the smallest possible number. Experience has shown that there are certain physical quantities from which all others can be derived, and thus we need only set up standards in these fundamental quantities—for this work mass, length and time are chosen and we keep a sample of each with which all others can be compared—but this is discussed further in subsequent paragraphs.

Physics is built up on two sorts of properties. The first are those which are measurable by *direct comparison* with the standard samples which we keep and are called *fundamental magnitudes*; hence our

fundamental magnitudes must also be mass, length and time. Secondly there are properties such as speed, of which we keep no standard; instead we derive speed by noting the time taken to cover a measured distance, i.e. by direct reference to two *fundamental magnitudes*; such quantities are called *derived quantities*, and examples are density, pressure, area, etc.

A derived quantity may usually be measured by a method that appeals directly to fundamental quantities or by methods involving the application of various physical laws. For example, the speed of a car may be found by reference to its speedometer; one type of speedometer uses a set of spinning magnets to deflect a pointer against the tension of a hair spring—if the physical laws of rotating magnets and hair springs are known, the pointer can be used to indicate speed. The method of measurement which appeals directly to fundamental quantities is called an *absolute measurement* of the derived quantity, and normally more reliance would be placed on this method than on one which involves the use of a number of physical laws.

1.2 Physical Dimensions

The basic quantities out of which our present quantities are developed are mass, length and time. These are sufficient to cover the quantities met in mechanics and general properties of matter, but to include the units of heat and electricity and magnetism other basic quantities must be added.

These three quantities were introduced as the most convenient at about the time of Newton. With certain restrictions, *any* three quantities could have been chosen as the basic ones; whether or not mass, length and time remain the most convenient quantities is a matter of doubt these days, and in the last few years various alternatives have been proposed. However, scientific method (and domestic life) are so strongly wedded to mass, length and time, and the suggested changes are so radical, that no alteration in the near future is likely.

These three quantities we call our *basic dimensions*, using for them the obvious symbols M, L, T. As new derived quantities are built up, it is possible to show how they are formed from fundamental magnitudes by writing a 'dimensional equation' indicating how the new property is related to mass, length and time. It is common to indicate such dimensional equations by the use of square brackets [ ], thus Equation (2) could be written

$$[\text{Speed}] = \frac{[L]}{[T]}$$

and this would be read as: 'The dimensions of speed are length divided by time.'

This use of the word 'dimension' should not be confused with the use

the carpenter makes of it when he says that the 'dimensions' of a box are 3 ft $\times$ 2 ft $\times$ 1 ft 6 in.

If an equation is to be a valid statement of a physical law, the dimensions of both sides of the equation must be the same. This is a difficult idea and it is treated again in more detail in Chapter 9, but it is of such value that the idea should be accepted at this stage. Whenever a new physical equation is developed, a check that the dimensions of both sides are identical indicates that the equation is probably correct.

If the physical quantities appearing on one side of an equation are divided by those on the other side, they will form a group which has no dimensions, it is therefore called a *dimensionless group* and is equated to a pure number. Further on in this book it will be seen that there are certain advantages to be obtained by manipulating dimensionless groups of quantities rather than the individual quantities.

1.3 Units

Having decided on the basic dimensions on which our Science is going to be built, it is then necessary to decide the unit in which each one of these quantities is to be measured. Let us discuss in particular the unit of length.

From time to time, many standards of length have been proposed, but by far the most important at the present time owes its origin to the reform of the systems of measurement undertaken in France under Napoleon.

The distance between two fine scratches made on a particular straight metal bar, preserved at Sèvres, near Paris, was defined as the *Metre,* and this standard length has been widely adopted all over the world.

The *Imperial Standard Yard* is a similar standard, defined by the distance between marks on another bar kept by the British Board of Trade, and there are many other standards of less importance.

The basic standards are:

Mass: a cylinder of platinum-iridium known as the International Prototype Kilogram (kg).

Length: a bar of platinum-iridium carrying two fine scratches. The distance between these scratches when the bar is at the temperature of melting ice is known as the International Prototype Metre (m).

Time: the average time between two successive passages of the Sun through the Meridian, known as the Mean Solar Day.

For scientific work these standards are rather large and we use:

Mass: the gram (gm) $= \frac{1}{1000}$ International Kilogram.

Length: the centimetre (cm) $= \frac{1}{100}$ Metre.

Time: Mean Solar Second (sec) $= \frac{1}{86400}$ Mean Solar Day.

This set of units is described as the 'centimetre-gram-second' or 'c.g.s. system'. In most English-speaking countries, domestic, commercial and, to some extent, engineering, dealings are conducted in a system based on other units, namely:

Mass: the pound (lb).

Length: the foot (ft) $= \frac{1}{3}$ Imperial Standard Yard.

Time: the second (sec).

This set of units is known as the 'foot-pound-second' or 'f.p.s. system'.

If we are to produce a consistent system of units building from the units of mass, length and time, it will be seen that once the size of each basic unit is chosen, the size of all other units derived from them is fixed. It is necessary then to choose the size of these original units so that all derived units are also of reasonable size. With one or two exceptions, the c.g.s. system of units fulfils this condition throughout General Physics; however, the units used in electricity and magnetism, if derived from the c.g.s. units, are not always of convenient size, some being as much as a thousand million times too small for practical use. In this branch of Physics it is found more convenient to take the metre, kilogram and second as the basic units, this is called the 'M.K.S. system' of units.

The consistent set of units which is used today had its beginning in the fourteenth century. Newton, at the end of the seventeenth century, was the first person systematically to build up the derived quantities from mass, length and time, although he did not set up units in which to measure these quantities. In 1822 Fourier pointed out that, since all derived quantities were expressed in terms of mass, length and time, the size of each one of these derived units was already fixed in terms of the centimetre, gram and second. The French Government had already decided to standardise the size of the metre, kilogram and second and in 1820 made the use of these units compulsory in France, but it was not until 1870 that an *international* agreement was reached to adopt the c.g.s. system of units for scientific use.

It should be noted that although all these systems of units are in existence, a true physical law can always be written in the dimensionless form and thus is independent of the units in which the constituent quantities of the law are measured; thus one set of laws can be developed which is applicable to the whole of Physics, whatever measuring system may be used.

1.4 Derived Units

Having fixed the basic dimensions and the size of the units in which they are going to be measured, we can proceed to derive some further quantities and their units.

Area

The simplest of these is *Area.* The area of a rectangle is defined as the product of its length and its breadth, or

$$\text{Area} = \text{length} \times \text{breadth}.$$

Now length and breadth are both measured in units of length, thus, using square brackets to indicate that we are considering only the dimensions of the quantities, we may write

$$[\text{Area}] = [L \times L]$$
$$= [L^2].$$

This equation should be read: 'The dimensions of Area are length squared.' If we are attempting to build a consistent set of units, we are not at liberty to choose at random the unit in which to measure area. Since the unit in which the dimension length is measured is the centimetre, then area, which has the dimensions $[L^2]$, must be measured in units of (cm^2) or square centimetres. In the f.p.s. system of units, the unit of area is the (ft^2) or square foot.

Volume

Similarly the volume of a rectangular parallelepiped is given by:

$$\text{Volume} = \text{length} \times \text{breadth} \times \text{thickness}$$

$$\text{and thus } [\text{Volume}] = [L \times L \times L]$$
$$= [L^3],$$

since thickness is also measured in units of length.

Thus the unit in which volume is measured must be (cm^3) or the cubic centimetre (cc) in the c.g.s. system and (ft^3) or the cubic foot in the f.p.s. system.

Density

Next we define the density of a substance as the mass of unit volume of that substance or:

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}.$$

$$\text{Thus } [\text{Density}] = \left[\frac{M}{L^3}\right]$$
$$= [ML^{-3}].$$

The c.g.s. unit of density then becomes the ($gm.cm^{-3}$) or gram per cubic centimetre, usually written gm/cc, while the f.p.s. unit is the ($lb.ft^{-3}$) or pound per cubic foot. The density of water is 1 $gm.cm^{-3}$, 62·5 $lb.ft^{-3}$ or 1000 $kg.m^{-3}$.

It will be noticed that several conventional ways are used above for writing the name of the unit. The index method ($gm.cm^{-3}$) is gaining popularity at the present time and will largely be used in this book.

Specific Gravity

The Specific Gravity of a substance can be defined as the density of that substance divided by the density of water, i.e. the ratio of two densities. Thus:

$$\begin{aligned}[\text{Specific Gravity}] &= \left[\frac{ML^{-3}}{ML^{-3}}\right]\\ &= [M^0L^0].\end{aligned}$$

Specific gravity has no dimensions and is not measured in terms of any units; it is in fact a pure number, and will retain the same value whatever system of units is in use. For example the density of alcohol in the c.g.s. system is 0·8 gm.cm^{-3}, and in the f.p.s. system is 50·0 lb.ft^{-3}. Its specific gravity, however, is always 0·8.

Several *dimensionless quantities* will be encountered in this work; usually, but not invariably, they are defined as the ratio of two similar quantities.

Speed

The next set of derived units introduces the dimension of time.

If a body is moving uniformly in a straight line, so that it covers equal distances in equal time intervals (however small we take those intervals), then we define its *Speed* as the distance covered in unit time or:

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}.$$

$$\begin{aligned}\text{This gives [Speed]} &= \left[\frac{L}{T}\right]\\ &= [LT^{-1}],\end{aligned}$$

and the c.g.s. unit of speed becomes the (cm.sec^{-1}) or centimetre per second. The f.p.s. unit of speed is the (ft.sec^{-1}) or the foot per second; other common units of speed are the mile per hour, mph, or the nautical mile per hour, called the *knot*.

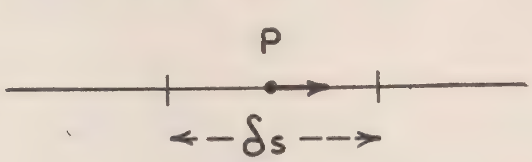


Fig. 1.1

If the speed of a particle is not constant, then we can find an average value for its speed at any point P (Fig. 1.1) by marking off a small length δs including the point P and finding the time taken by the particle to traverse this distance. If it takes a time δt, then its average speed, $\bar{v}$, over the distance δs is given by:

$$\bar{v} = \delta s/\delta t.$$

δs can be made as small as we please, and in fact can be made so small that the speed remains approximately constant within its limits. If δs tends to zero, then the average speed becomes a closer and closer approximation to the actual speed of the particle at P. If this speed is v, then using the notation of the calculus, we can write:

$$v = \lim_{\delta s \to 0} \delta s/\delta t$$

$$\text{or } v = ds/dt \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

which can be interpreted as 'speed is the rate of change of distance with time'.

Acceleration

If the speed of a particle moving in a straight line is changing uniformly, so that the speed increases or decreases by equal amounts in equal intervals of time, then the particle is *accelerating* or *decelerating*.

The acceleration of the particle is defined as the change in speed occurring in unit time, or

$$\text{Acceleration} = (\text{Change in Speed}) \div (\text{Time})$$

$$\text{Thus } [\text{Acceleration}] = \left[\frac{LT^{-1}}{T}\right] = [LT^{-2}].$$

The c.g.s. unit in which acceleration is measured is the (cm.sec^{-2}); this is variously written as 'cm per sec per sec' or 'cm per sec^2', neither of which is very suitable. Similarly the f.p.s. unit of acceleration, the (ft.sec^{-2}), is often called the 'foot per sec per sec' or 'foot per second squared'.

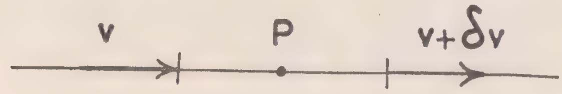


Fig. 1.2.

If the acceleration is not constant, we can find its value at any point P as before. If δv is the change in speed over a time interval δt during which the particle passes P (Fig. 1.2), then the average acceleration over the interval is given by $\delta v/\delta t$; as the interval becomes smaller and smaller, this average becomes closer and closer to the true value of the acceleration at P, thus, using the symbol a for acceleration, we have:

$$a = \lim_{\delta t \to 0} \delta v/\delta t$$

$$= dv/dt \quad . \quad . \quad . \quad . \qquad (4)$$

Also, since $v = ds/dt$,

$$a = \frac{d}{dt}(v)$$

$$= \frac{d}{dt}\left(\frac{ds}{dt}\right)$$

$$= \frac{d^2s}{dt^2} \quad . \quad . \quad . \quad . \quad (5)$$

Acceleration due to Gravity

It is found experimentally that all bodies at the same place near the surface of the Earth fall freely with the same value of acceleration. This is called 'the acceleration due to gravity' and is denoted by the symbol g.

$$g = 980 \text{ cm.sec}^{-2}$$
$$\text{or } g = 32{\cdot}2 \text{ ft.sec}^{-2}$$

The value is not the same at all points on the Earth's surface and variations of about 0·5% are noticed in the figures given above.

The value of g is sometimes used as a unit of acceleration, thus an acceleration of 96·6 ft.sec^{-2} can be described as an acceleration of $3g$.

1.5 Rectilinear Motion

From Equations (1) to (5) we can derive a set of formulæ useful in the solution of problems in uniform motion.

Thus, for a particle moving at constant speed, from Equation (1) we have:

$$s = vt \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For a particle moving with constant acceleration, we have:

Change in Speed = Acceleration × Time

so that if its speed has initially the value u and finally the value v:

$$v - u = at$$
$$\text{or } v = u + at \quad . \quad . \quad . \quad . \quad (7)$$

The distance moved in this case is given by:

$$\text{Initial velocity} = u$$
$$\text{Final velocity} = v$$
$$\therefore \text{Average velocity} = \frac{u + v}{2}$$

and distance moved in time t is given by:

$$s = \frac{u + v}{2} . t$$

Substituting for v from Equation (7) gives:

$$s = \frac{u + \{u + at\}}{2} . t$$

$$= ut + \tfrac{1}{2}at^2 \quad . \quad . \quad . \quad . \quad (8)$$

The student who is acquainted with the calculus could derive this equation by writing ds/dt for v in Equation (7) and then integrating once.

Next eliminate t between Equations (7) and (8).

From (7) $$t = \frac{v - u}{a}.$$

Substituting for t in (8) gives:

$$s = u\frac{(v - u)}{a} + a\frac{(v - u)^2}{2a^2}$$

or $$2as = -u^2 + v^2,$$

giving $$v^2 = u^2 + 2as \quad . \quad . \quad . \quad . \quad (9)$$

Equations (7), (8) and (9) are the common ones used in the solution of problems in uniform motion. It will be noticed, however, that they contain five variables, u, v, s, a, t, four of which appear in any one equation; thus s is missing from (7), v from (8) and t from (9).

It is instructive and useful to construct the two remaining equations, deficient in u and a respectively.

Eliminating u between (7) and (8) gives:

$$s = vt - \tfrac{1}{2}at^2. \quad . \quad . \quad . \quad . \quad (10)$$

and then the acceleration can be eliminated between (8) and (10) giving:

$$s = \frac{u + v}{2}\,.\,t \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Note that $\frac{1}{2}(u + v)$ is the average speed over the time interval t.

Since each of Equations (7) to (11) contains four variables, any problem can be solved which gives the values of three variables.

Graphical Treatment of Rectilinear Motion

If the distance travelled by a body is plotted in rectangular coordinates against the time taken for the journey, the resulting graph is called a 'space-time' curve of the motion. The gradient or slope of the curve at any point is a measure of the speed (ds/dt) at that point or instant.

A further curve of speed against time can thus be constructed and the slope of this curve gives the acceleration of the body.

The area under any portion of the speed-time curve is a measure of the product (speed $\times$ time) and thus gives the distance travelled by the body during any time interval. This is true whether the acceleration is constant or varying, and the graphical method is often the only simple way of treating this latter case.

1.6 Vector and Scalar Quantities

So far in this chapter we have ignored one very important difference between length (also some units derived from length) and mass and time. Length frequently has the idea of *direction* associated with it, but the others never have. Thus to travel 2 miles north from a given starting-point produces a very different result from travelling 2 miles east from the same starting-point. The *direction* in which we travel is obviously just as important as the *distance*.

Most of the important quantities of Elementary Physics can be divided into *vectors* and *scalars*. Vector quantities have direction, all others are called *Scalar Quantities*. Length can be a vector, mass and time are scalars.

In some particular circumstances the direction of a vector quantity is not important, as in the foregoing work where we have been dealing with motion in a straight line; in this case all distances are measured in the same direction, and they *can be treated* as scalar quantities. The reverse effect, however, does not occur and scalar quantities never assume a vector nature.

In this connection, a peculiar distinction is made in the case of speed. When we are concerned with the magnitude of the quantity only, and therefore are treating it as a scalar, then the name 'speed' is retained, but when we are also concerned with the direction of the quantity and are treating it as a vector, the quantity is renamed *Velocity*. It is not common to change the name of a unit in this fashion.

Of the quantities that have been met so far in this chapter, length, velocity and acceleration all have direction associated with them and so can be vectors.

Area can also be treated as a vector quantity, since an area is usually a plane which has a fixed direction in space. The direction of the plane and hence of the area vector is specified by the direction of a line drawn perpendicular to the plane.

Scalar and Vector Addition

The addition of two scalars merely necessitates the arithmetical addition of the numbers representing the magnitude of the quantities; thus to add 17 gm to 23 gm, we add 17 to 23, giving 40 gm as the answer. Notice that both scalars must be like quantities, i.e. measured in the same units, for we cannot sensibly perform an addition of such unlike things as:

$$17 \text{ gm} + 23 \text{ sec.}$$

The addition of two vector quantities cannot be carried out by arithmetical methods, since these processes take no account of the direction of the vector. The simple vectors, length, velocity, acceleration, and more to be defined later, can all be represented by scale drawings, an arrowhead being used to show the sense of the vector. With the

help of such drawings vectors can be added by geometric processes.

The vectors to be added are drawn in the correct direction and to scale, end to end, and with all the arrowheads pointing in the same direction around the figure; the vector required to close the figure but with its arrowhead opposing all the others is then the *vector sum.*

This addition process can be applied to any number of vectors, they can be taken in any order (as long as the arrowheads 'follow on'), and the resulting figure may cross over itself.

In the vast majority of problems which the reader is likely to encounter, only two vectors at a time have to be added or *compounded.* These two vectors with their vector sum or *resultant* always form a triangle, hence the vector addition process is often called 'completing the triangle'.

The graphical method of vector addition gives results which are accurate enough for most purposes. If a more exact result is needed the lengths of the sides of the triangle and its angles may be calculated trigonometrically; for details of this method the reader should consult the Appendix of *Advanced Level Applied Mathematics*, by C. G. Lambe, B.A., Ph.D., published by the English Universities Press, Ltd.

Vector Subtraction

The negative of a given vector is of the same length but points in the opposite direction; thus if A and B are vectors, the vector difference is given by:

$$A - B = A + (-B),$$

hence to subtract B from A, we add to A the negative of B, i.e. B reversed in direction.

Resolution of Vectors

Just as frequently as vectors are compounded, so the reverse process

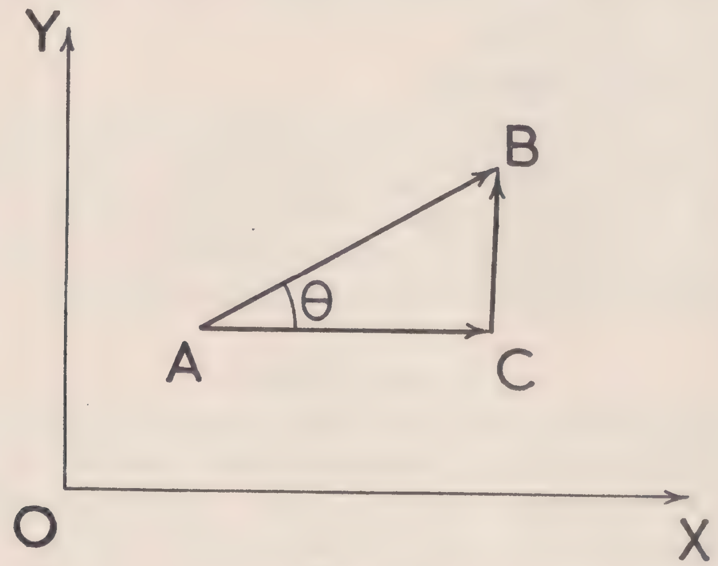


Fig. 1.3

has to be carried out, i.e. it is required to find a pair of vectors which, when compounded, would be equal to a given vector. Such a process is called *resolving* a vector into components.

It is common to resolve a vector into two components in specified directions, usually perpendicular to each other; thus the vector AB (Fig. 1.3), resolved into components parallel to OX and OY, becomes AC and CB where:

$$AC = AB \cos\theta$$
$$\text{and } CB = AB \sin\theta.$$

1.7 Relative Motion

The phenomenon of relative motion is probably most apparent when travelling on a railway.

Everyone waiting in a train standing at a station has experienced a sensation of forward movement only to find that the impression has been produced by a train pulling out in the opposite direction on a neighbouring track. The observer, being accustomed in a moving train to see the scenery moving backwards past the window, looks out of the carriage and sees another train moving backwards; this he promptly and erroneously interprets as due to his own forward motion.

This example serves to show that motion may be interpreted in several ways according to the circumstances of the observer. Here the confusion has arisen since the observer incorrectly assumed the other train to be at rest, and deduced his own motion from this false assumption.

The alternative problem is often encountered. In this case the true velocity of the observer and of the object he observes is known, and the velocity of the object relative to the observer is required; that is, the velocity the observer would ascribe to the object if he incorrectly assumed himself to be at rest.

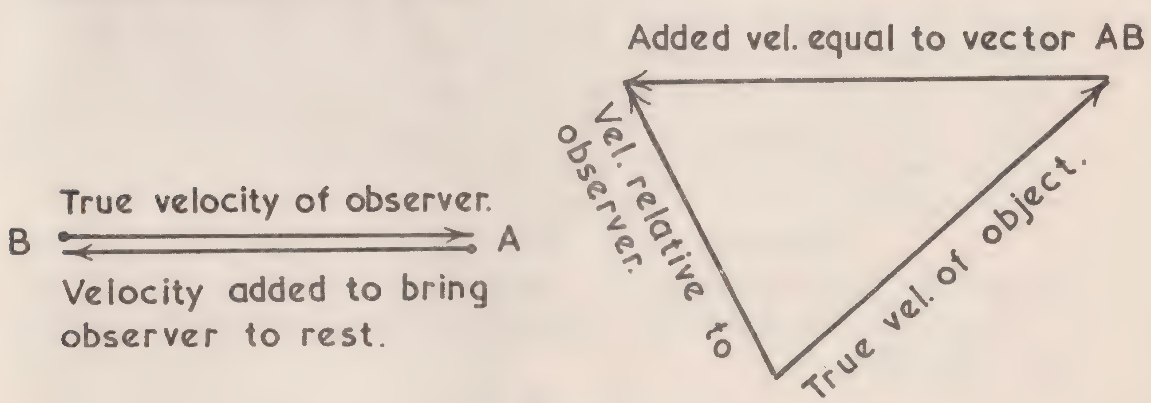


Fig. 1.4

Since the observer imagines himself at rest, add to his velocity a vector equal and opposite in direction, thus bringing him to rest (Fig. 1.4). To preserve the same relative motion between observer and ob-

ject, add a similar vector to the velocity of the object; the vector sum of these two velocities is the velocity of the object *relative to the observer* since he is now at rest. In practice, this merely means that we have to add to the motion of the object the negative of the motion of the observer.

SUMMARY OF UNITS INTRODUCED IN THIS CHAPTER

Quantity	*Dimensions*	*c.g.s. unit*	*f.p.s. unit*	*M.K.S. unit*	*Gravitational unit*
Mass	M	gram (gm)	pound (lb)	kilogram (kg)	—
Length	L	centimetre (cm)	foot (ft)	metre (m)	—
Time	T	second (sec)	second (sec)	second (sec)	—
Area	L^2	square cm (cm^2)	square ft(ft^2)	square m(m^2)	—
Volume	L^3	cubic cm (cm^3)	cubic ft(ft^3)	cubic m(m^3)	—
Density	ML^{-3}	gm per cubic cm ($gm.cm^{-3}$)	lb per cubic ft ($lb.ft^{-3}$)	kg per cubic m ($kg.m^{-3}$)	—
Specific gravity	Zero	Pure Number			
Speed or Velocity	LT^{-1}	cm per sec ($cm.sec^{-1}$)	ft per sec ($ft.sec^{-1}$)	m per sec ($m.sec^{-1}$)	—
Acceleration	LT^{-2}	$cm.sec^{-2}$	$ft.sec^{-2}$	$m.sec^{-2}$	$g = 980$ $cm.sec^{-2}$ or 32·2 $ft.sec^{-2}$

EXERCISES 1

1. A jet aircraft leaves London at 9.0 a.m. and flies due west against a 100-m.p.h. wind to Gander, Newfoundland. It arrives at 11.0 a.m. (local time). If it leaves Gander at 12 noon (local time), when will it arrive back in London? Assume that London and Gander are 2000 miles apart, and that they are at the same latitude, λ, where $\cos \lambda = 2/\pi$. (Earth's radius = 4000 miles.)
(Cambridge Univ. Schol., King's College Group (part).)

2. Show that a body travelling in a straight line, initially moving with a velocity u and acceleration f, traverses a distance S in time t given by the expression:
$$S = ut + \tfrac{1}{2}\, ft^2.$$
How would you test this relation experimentally for a body starting from rest and moving under constant acceleration?
A body falls freely from the top of a cliff, and during the last second it falls $\frac{11}{36}$ of the whole height. What is the height (in feet) of the cliff?
(Cambridge H.S.C.)

3. Prove that for a particle moving from rest with uniform acceleration the area under the velocity-time graph gives the distance covered by the particle.

   A train moves from rest to rest between two stations in 18 minutes. For the first two minutes it moves with a uniform acceleration of $\frac{1}{2}$ ft. per sec. per sec. and thereafter its speed is constant until it is brought to rest by a constant braking force acting for one minute. Sketch the velocity-time graph of the motion, and hence or otherwise find the distance between the two stations. (London Univ. Inter. B.Sc.)

4. A car moves from rest with an acceleration which decreases uniformly with the time, its initial value being 1·8 ft./sec.^{2}, and its rate of decrease such that the car would attain its maximum velocity in 60 seconds. After running for 40 seconds the engine is shut off and the brakes are applied, causing a retardation whose numerical value increases uniformly with the time from an initial value of 0·8 ft./sec.^{2} The car is brought to rest in 32 seconds after the application of the brakes.

   Draw the acceleration-time graph, finding the maximum velocity actually attained and the final retardation. Find the velocity 20 seconds after starting and 16 seconds after the application of the brakes.

   Sketch the velocity-time graph. (London Univ. Inter. B.Sc.)

5. Two trains, of lengths 300 ft. and 342 ft., moving in opposite directions along parallel tracks, meet when their speeds are 30 m.p.h. and 45 m.p.h. respectively. If the first train is being accelerated at 1 ft./sec.^{2} and the second is being retarded at 2 ft./sec.^{2}, calculate the time in seconds taken by the trains to pass each other completely. (London Univ. Inter. B.Sc.)

6. A balloon rises from rest on the ground with constant acceleration $\frac{1}{8}g$. A stone is dropped when the balloon has risen to a height H feet. Find the time taken by the stone to reach the ground. If a second stone is dropped t seconds after the first one, find the distance between the stones t seconds after the second stone is dropped, assuming that the first stone had not reached the ground at that time. (London Univ. Inter. B.Sc.)

7. Explain the rule for finding the velocity of a particle relative to another which is also moving. Illustrate your answer by an example.

   A river 220 yds. wide flows at 4 m.p.h., and a man who can row at 5 m.p.h. in still water sets off from one bank to a point on the other side directly opposite his starting-point. Find the time he takes if his path is always at right angles to the banks.

   Find also the shortest possible time in which he can travel from one bank to the other. (Oxford H.S.C.)

8. Explain what is meant by the relative velocity of one moving object with respect to another.

   A ship A is moving eastward with a speed of 15 knots and another ship, B, at a given instant 10 nautical miles east of A, is moving

southwards with a speed of 20 knots. How long after this instant will the ships be nearest to each other, how far apart will they be then, and in what direction will B be sighted from A?

(Cambridge H.S.C.)

9. Two roads OA, OB cross at O; OA leads east and OB leads N. 30° E. A man on OA, 4 miles east of O, and walking towards O at $3\frac{1}{2}$ m.p.h., observes a vehicle on OB bearing N. 15° E. One half-hour later he observes its bearing is N. 15° W. Find, by drawing or calculation, (i) the speed of the vehicle, assumed constant, (ii) the shortest distance between the man and the vehicle, assuming they continue to move with the above constant speeds. (London Univ. Inter. B.Sc.)

10. A ship, steaming on a straight course at 20 miles per hour, is timed to pass a pier at 3 p.m., the nearest point of the ship's course being distant $\frac{1}{2}$ mile from the pier. A boat, which is rowed at 4 miles per hour, leaves the pier so as to intercept the ship.

(i) If the boat starts when the ship is 3 miles away from the pier, find, graphically or otherwise, at what angle to the ship's course the boat should be steered.

(ii) Find the latest time at which the boat could start.

(London Univ. Inter. B.Sc.)

11. When cycling at 12 m.p.h. along a road which runs SSW., a cyclist finds that the wind appears to come from the WSW. On turning into a road which runs SE. and cycling at 10 m.p.h., he finds that the wind appears to come from the E. Find the speed and direction of the wind. (London Univ. Inter. B.Sc.)

CHAPTER 2

NEWTON'S LAWS OF MOTION

2.1 Introduction

Most of the theories that are used today concerning the motion of bodies are derived from three basic statements, propounded by Newton and now known as his Laws of Motion. These laws resulted from the gathering together of various *experimental* facts, many of which had been noted or discovered before Newton's time. The work which leads up to Newton's Laws is discussed in a later chapter; at present we shall merely state the laws and then examine their consequences.

Newton's Laws of Motion may be stated in the following form:

I. Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.

II. The rate of change of momentum is proportional to the motive force impressed and takes place in the direction in which that force acts.

III. To every action there is always opposed an equal reaction; or the mutual actions of two bodies upon each other are always equal and oppositely directed.

Newton's Laws introduce two new quantities, momentum and force which must now be examined more closely.

2.2 Momentum

Momentum is the name used for the product (mass $\times$ velocity).

$$\begin{aligned}\text{Thus [Momentum]} &= [\text{Mass} \times \text{Velocity}]\\ &= [M \times LT^{-1}]\\ &= [MLT^{-1}]\end{aligned}$$

and from this it is seen that the unit in which momentum is measured in the c.g.s. system is the gm.cm.sec^{-1}, and in the f.p.s. system the lb.ft.sec^{-1}. The c.g.s unit is also called the dyne.sec, for a reason which will appear later in this chapter.

Momentum is the product of mass, which is a scalar quantity, and velocity, which is a vector. The product of a vector and a scalar is always a vector, and thus momentum is a vector quantity and must be treated according to the rules of vector addition.

2.3 Force

If Newton's second law is expressed as an equation, then:

$$\frac{d}{dt}(mv) \propto f$$

where m is the mass of a body, v its velocity and f the impressed force.

$$\text{This gives } f = k\frac{d}{dt}(mv) \quad . \quad . \quad . \quad . \quad (1)$$

where k is a constant of proportionality.

Now the mass of a body is usually constant, thus we can write it outside the differential coefficient, and Equation (1) becomes

$$\begin{aligned} f &= km\, dv/dt \\ &= kma \end{aligned}$$

since dv/dt is the acceleration of the body.

This equation relates the new quantity, force, to two which are already known, i.e. mass and acceleration, and can be used to define the size of the unit in which force is to be measured and the magnitude of the constant k. When developing a consistent set of units, the size of the unit of force is so chosen that the value of k in the equation above becomes equal to unity. This will be so if one unit of force is the magnitude of the force which, when acting on a mass of 1 gm, gives it an acceleration of 1 cm.sec^{-2}; thus f, m and a in the above equation all take the value unity, making k equal to one. Hence, *if force is measured in these units*, then it will be related to mass and velocity by the equation

$$f = ma \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which the constant k has become equal to unity and so disappeared. The dimensions of force are given by the equation

$$\begin{aligned} [\text{Force}] &= [\text{Mass} \times \text{Acceleration}] \\ &= [M \times LT^{-2}] \\ &= [MLT^{-2}]. \end{aligned}$$

The unit of force in the c.g.s. system of units is thus the gm.cm.sec^{-2} which, for brevity, is known as the *dyne*, and in the f.p.s. system it is the lb.ft.sec^{-2}, also called the *poundal*. The M.K.S. unit of force is the kg.m.sec^{-2} which is renamed the *newton*

$$\begin{aligned} 1 \text{ newton} &= 10^5 \text{ dynes} \\ &= 10^5/980 \text{ gm-wt (see page 21.)} \\ &\simeq 100 \text{ gm-wt.} \end{aligned}$$

Force is a vector quantity, being the product of a vector and a scalar; indeed our experience of a force tells us very strongly that it is applied in a given *direction* and hence must be a vector; this fact is also stressed in Newton's second law.

2.4 Mass and Weight

It is now possible to obtain a further insight into the meaning of mass. From Equation (2) it is seen that if the mass of a particle is small, a given force applied to it will produce a large acceleration, and that the acceleration becomes smaller as the mass increases—in fact they are inversely proportional to each other. The mass of a body appears then to be the property which makes it resist an acceleration; it is the property of the body which is sometimes called, rather loosely, its *Inertia.*

The two descriptions of mass—quantity of matter in the body and quantity of inertia possessed by the body—should both be remembered. Together they give quite a good illustration of the meaning of mass.

To connect the ideas of mass and weight, the important fact is the experimental observation that all bodies falling freely near the surface of the Earth have the same acceleration—g, the acceleration due to gravity.

Now the application of a force is needed to produce an acceleration, thus a falling body must be experiencing a force directed towards the Earth to give it the acceleration.

If the body is of mass m, and it falls with an acceleration g, then the force acting on it must be given by:

$$f = mg.$$

This especial force is called the *weight* of the body, thus:

$$W = mg \qquad (3)$$

and the weight of a body is proportional to its mass, the ratio between the two being the acceleration due to gravity.

The position can be summarised as follows:

Mass	*Weight*
Quantity of matter in a body.	The downward pull which is experienced when holding a body in the hand.
Inertia of a body which makes it resist acceleration.	Force which makes a body accelerate towards the centre of the Earth.
Depends only on the material and size of a body.	Depends also on the presence of the Earth.
Constant.	Varies from place to place.

Note that the weight of a body is a *force* and must be measured in dynes or poundals. This causes endless confusion, since in every laboratory

there are many boxes of small brass cylinders which are always known as 'weights', and are used for 'weighing' other objects; stamped on these weights, however, appear such legends as 100 *gm*, and, having weighed a calorimeter, one is very tempted to record 'Weight of Calorimeter = 100 gm'. But the gram is the unit of *mass*, i.e. on the 'weight' is stamped its mass, NOT its weight. This is because g, the acceleration due to gravity, is not exactly constant, but varies slightly from place to place on the Earth's surface, and decreases with height. Although the mass of a body remains constant, its weight varies as gravity changes; we obviously cannot stamp on a body a quantity which has no fixed value and so we resort to stamping on a 'weight' its unvarying mass. The user has to remember (or just as often to forget?) that if he wishes to know the weight of a 'weight', then he must multiply the mass stamped on it by the appropriate value of the acceleration due to gravity.

2.5 Force versus Mass as a Fundamental Quantity

It will be remembered that in Chapter 1 it was found difficult to justify the choice of mass as a fundamental unit; reasons for this choice will now perhaps be more apparent. Whilst it is difficult to define mass, quantity of matter and inertia have very real meanings for most of us, also a sample mass can very easily be made and stored.

The effect of mass which we most often experience, however, is its weight, and there might be some reason for taking weight (or force) as a fundamental quantity; but force is a more elusive quantity than mass and it is much more difficult to reproduce a 'standard force', whereas there is little difficulty in making faithful copies of a standard mass.

For these reasons, and as a result of the experience of the earlier physicists, mass was finally chosen as the third basic quantity in preference to force for scientific work.

2.6 Gravitational Units of Force

Since weight is the most common of all forces, it has become customary to compare forces with weights, thus the magnitude of a particular force is said to be the same as the weight of a five-pound mass or, more commonly, the magnitude of the force is the same as five pounds weight. These statements are both quite true and self-consistent if by 'five pounds weight' we mean 'the weight of a five-pound mass'.

By convention the phrase 'pounds weight', written as lb-wt, means 'the weight of a one-pound mass'. The lb-wt, and its counterpart the gm-wt, are thus forces and can be used as units of force.

$$\text{Obviously } M \text{ lb-wt} = M \times g \text{ poundals} \quad (g = 32{\cdot}2 \text{ ft.sec}^{-2}),$$
$$\text{and } m \text{ gm-wt} = mg \text{ dynes} \quad (g = 980 \text{ cm.sec}^{-2}).$$

Thus 5 lb-wt is 161 poundals, etc.

The gm-wt and the lb-wt are known as gravitational units of force, since their value depends on the local value of g.

Thus 1 gm-wt = 981 dynes in England

but 1 gm-wt = 983 dynes at the North Pole,

since the values of g at these two places are 981 and 983 $cm.sec^{-2}$ respectively.

A danger lies in the use of gravitational units, for the statement 'a force of five pounds weight' can so easily be abbreviated to 'a force of five pounds'. This statement is quite wrong, since a force cannot be measured in terms of the pound, which is the unit of mass. The use of such slipshod phraseology leads to many errors in calculation, for the student who equates a weight to a mass will arrive at an answer numerically incorrect owing to the absence of the factor 32·2 or 981 in the calculation.

Nevertheless, the student should be prepared to meet such misstatements, even in examination papers, and should supply the deficiency himself, as in the following example.

Example 1. *A man who weighs* 150 *lb is sliding down a rope with a velocity of* 5 *ft per sec. By gripping the rope, he exerts an upward force on himself of* 150 *lb. What is his velocity after* 3 *seconds? If he suddenly slackens his grip so that the upward force is reduced to* 90 *lb what is the value of his acceleration?*

This example should read 'A man of mass 150 lb . . . exerts an upward force on himself of 150 lb-wt. . . . force is reduced to 90 lb-wt. . . .,' etc.

2.7 Impulse

Equation (2) was developed from the mathematical statement of Newton's second law, viz:

$$f = \frac{d}{dt}(mv).$$

This can be rewritten as

$$f\,dt = d(mv).$$

If the force acts from time 0 to time t, and during this interval the momentum changes from $(mv)_1$ to $(mv)_2$, then

$$\int_0^t f\,dt = \int_{(mv)_1}^{(mv)_2} d(mv)$$

$$= (mv)_2 - (mv)_1 \quad . \quad . \quad . \quad (4)$$

The quantity $\int_0^t f\,dt$ is called the *Impulse* of a force and, from Equation (4), is equal to the change of momentum produced by the force.

Thus:

Impulse = Change in Momentum.

Since impulse is only a change in momentum, it will be a vector

quantity, will have the same dimensions as momentum, and will be measured in the same units, i.e. gm.cm.sec^{-1}; but since impulse is also the product of a force and a time it can be measured in dyne.sec and this unit can also be used for momentum. The two units are, of course, identical.

If the force remains constant, then

$$\int_0^t f\,dt = ft$$

or the impulse of a constant force is equal to (force $\times$ time).

This idea is of great use in solving problems in which the force needed to bring a moving body to rest has to be calculated.

Example 2. *A jet of water* 1 *inch in diameter, and having a velocity of* 20 *ft.sec*$^{-1}$, *hits a wall without rebounding. Find the force exerted on the wall.*

In t seconds a stream of water $20t$ feet long hits the wall.

$$\text{Volume of this stream} = 20t \times 12 \times \pi \times (\tfrac{1}{2})^2 \text{ cu in.}$$

$$\text{But density of water} = 63 \text{ lb per cubic foot.}$$

$$\text{Mass of waters tream} = \frac{20t \times 12 \times \pi \times (\frac{1}{2})^2}{12^3} \times 63 \text{ lb}$$

$$\text{Momentum lost by water} = \frac{20t \times 12 \times \pi \times (\frac{1}{2})^2 \times 63 \times 20}{12^3} \text{ poundalsec.}$$

If the force exerted on the wall is f, then the impulse is ft and

$$ft = \frac{20t \times 12 \times \pi \times (\frac{1}{2})^2 \times 63 \times 20}{12^3}$$

$$\text{or } f = \frac{20^2 \times \pi \times 63}{12^2 \times 2^2} \text{ poundals}$$

$$= \frac{20^2 \times \pi \times 63}{12^2 \times 2^2 \times 32{\cdot}2} \text{ lb-wt}$$

$$= 4{\cdot}26 \text{ lb-wt}$$

Example 3. *Water is flowing at* 20 *ft.sec*$^{-1}$, *around a right-angle elbow bend in a* 1-*in. diameter pipe. Find the force on the pipe.*

Water enters the pipe in the direction A and leaves in the direction B.

Remembering that momentum is a vector quantity, all the momentum in the direction A possessed by the water on entering the pipe vanishes from the system, but an equal amount appears in the direction B.

The momentum lost in the direction A will be numerically the same as in the previous example, as also will be the impulse needed to bring about this loss of momentum. Now the impulse must be directed against the stream of water and acts on the water, hence the reaction of the water on the pipe will be in the same direction as the stream, thus the ingoing stream will exert a force F_1 on the pipe as shown in the diagram, where $F_1 = 4{\cdot}26$ lb-wt.

Similarly, to create momentum in the direction B, an impulse must be applied with the stream and the reaction on the pipe will be against the stream, F_2 in the diagram, where $F_2 = F_1$.

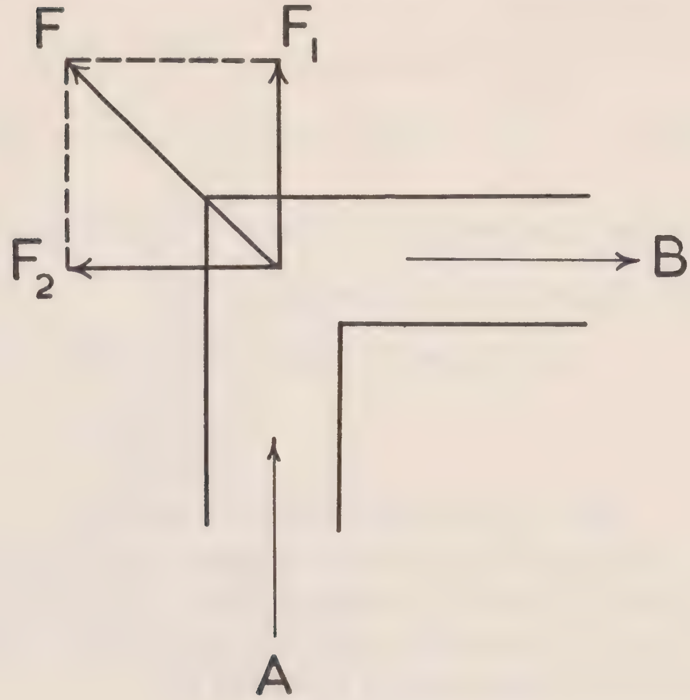


Fig. 2.1

The resultant force F is thus equal to $\sqrt{2}F_1$, i.e. 6 lb-wt, and acts at 45° to either arm of the elbow.

2.8 Conservation of Momentum

If two bodies act on each other so that they both accelerate, as for example two magnetised bodies attracting each other, then by Newton's third law they experience at all times equal but oppositely directed forces. These forces must act on both bodies for the same time and thus the two bodies will receive equal but oppositely directed impulses. The momentum of each body will therefore be changed by an equal amount, but in the opposite direction.

Since momentum is a vector quantity, the result of this interaction between the two bodies produces no change in the total momentum of the system, i.e. one gains as much momentum in a particular direction as the other loses.

This is an example of the *Principle of Conservation of Momentum* which may be stated in the form:

> In any system, the total momentum of all the bodies in a given direction is not changed by any interaction between.

The use of this principle is best illustrated by an example.

Example 4. *An aeroplane weighing* 4000 *lb is travelling at* 420 *mph when it fires a burst of* 40 *rounds from each of four* 20-*mm cannon. If each round weighs* $\frac{1}{4}$ *lb and has a muzzle velocity of* 2750 *ft.sec*$^{-1}$, *estimate the speed of the plane after firing the burst.*

$$\text{Gain in momentum by 1 round} = \tfrac{1}{4} \times 2750 \text{ lb.ft.sec}^{-1}.$$

But 160 rounds are fired,

$$\text{Thus total gain in momentum} = \frac{2750 \times 160}{4} \text{ lb.ft.sec}^{-1}.$$

If momentum is to be conserved, plane must lose an equal amount of momentum,

$$\text{Thus loss of momentum by plane} = \frac{2750 \times 160}{4} \text{ lb.ft.sec}^{-1}.$$

Ignoring loss of mass by plane as bullets are fired,

$$\begin{aligned} \text{loss of speed by plane} &= \frac{2750 \times 160}{4000 \times 4} \text{ ft.sec}^{-1} \\ &= 27\tfrac{1}{2} \text{ ft.sec}^{-1} \\ &= 18\tfrac{3}{4} \text{ mph.} \\ \text{Speed of plane after firing} &= 401\tfrac{1}{4} \text{ mph.} \end{aligned}$$

The truth of the principle of conservation of momentum can be demonstrated using a ballistic balance. Since this principle has been derived from Newton's third law, evidence of the truth of the principle must be regarded as substantiating the third law.

2.9 Rocket and Jet Propulsion

Both a rocket and a jet aeroplane propel themselves by ejecting a stream of gas from the tail.

Consider the start of the flight of a rocket; at first it is standing still with the fuel not yet ignited, thus the momentum of the system is zero. When the fuel begins to burn, the stream of exhaust gases acquires a backward momentum and so the rocket assumes an equal momentum in the forward direction in order to maintain the total momentum of the system at zero. Notice that this does not rely on the presence of an atmosphere—the rocket reacts on its own exhaust gases and would perform equally well in a vacuum or free space.

2.10 Work

If a force acts on a body, and as a result the body moves in the direction of the force, then the force is said to do *work* on the body.

The amount of the work done is measured by the product (force × distance moved in the direction of the force), or

$$W = f \times d \qquad \qquad (5)$$

The dimensions of work are given by

$$\begin{aligned} [W] &= [f \times d] \\ &= [MLT^{-2} \times L] \\ &= [ML^2T^{-2}] \end{aligned}$$

and thus the c.g.s. unit of work is the $\text{gm.cm.}^2\text{sec}^{-2}$, which is renamed the *erg*. One erg is the amount of work done when a force of 1 dyne causes a motion of 1 cm in the direction of the force, hence it is sometimes called the dyne.cm.

The erg is a very small unit in which to measure our efforts and we find it convenient to define a larger unit for practical purposes. This is the *joule* which is equal to 10^7 ergs; it is often called the *practical unit*, whilst the erg is called the *absolute unit.*

The M.K.S. unit of work is the work done when a force of 1 newton causes a movement of 1 metre. This unit could be called the newton. metre but: 1 newton.metre $= 10^5$ dynes $\times$ 100 cm

$$\begin{aligned} &= 10^7 \text{ dyne.cm} \\ &= 10^7 \text{ ergs} \\ &= 1 \text{ joule.} \end{aligned}$$

Thus the joule is the M.K.S. unit of work.

The equivalent f.p.s. unit of work is the lb.ft.^{2}sec^{-2}. This is the work done when a force of 1 poundal causes a movement of 1 ft in the direction of the force. The unit is much more commonly known as the ft.poundal.

The gravitational units of work are the cm.gm-wt (equal to 981 ergs) and the ft.lb-wt (equal to 32·2 ft.poundals), this latter unit is usually called the ft.lb.

Experience indicates that work is just as laborious no matter in which direction we go to perform it, i.e. there is no especial direction associated with the magnitude of work and thus we expect it to be a scalar quantity.

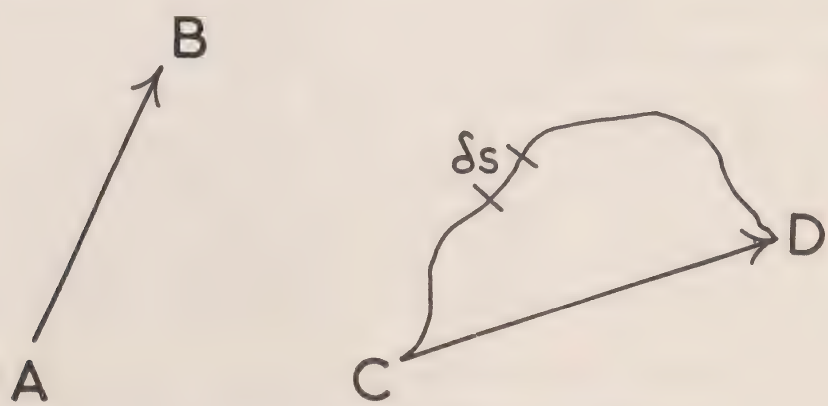


Fig. 2.2

If a force in the direction AB (Fig. 2.2), causes a body to move from C to D by any path such as that shown, the work done along any element δs of this path will be given by the product of the force and the component of δs in the direction of the force. The total work will be found by summing for all elements of the path, i.e.:

Work = Force $\times$ Σ components of δs along the direction AB, (the symbol Σ means 'the sum of'). The sum of the components of δs along the direction AB will be the component of CD along the direction AB and thus the work done between two points will be the same whatever the path chosen between them.

If the force varies both in magnitude and direction from point to point, the previous result still holds; if it did not, then between two points A and B (Fig. 2.3) it would be possible to find two paths ACB and $AC'B$ along which different amounts of work have to be done.

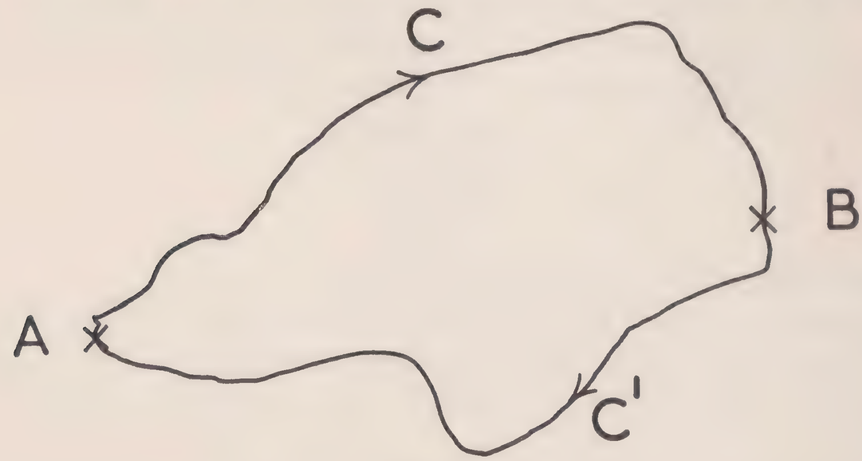


Fig. 2.3

Suppose that the work done along $AC'B$ is the greater and imagine a body moved from A to B via C and back from B to A via C'. Then the work expended on the outward journey will be more than regained on the return journey and an inexhaustible supply of work is available merely by going round and round the cycle. This is contrary to our experience and we conclude that, even in this case, the same amount of work is done when travelling between the same two points by any path.

This idea is of great value, especially when solving problems in magnetism and electrostatics. For example, if the work done in moving a magnetic pole between two points in a magnetic field has to be calculated, then the path which offers the easiest calculation can be chosen—this is usually the path that coincides in direction with the force at all points, as would happen if a magnetic pole were moved along a line of force.

If the force and the path coincide in direction at all points the force may still vary from point to point along the path. In this case, split the motion into small intervals δs, over each one of which the force will be practically constant. The work δW done in this small interval is then given by

$$\delta W = f \, . \, \delta s$$

and the total work in a displacement s is given by

$$W = \int_0^s f \, ds \qquad . \qquad . \qquad . \qquad . \qquad (6)$$

If the law relating the force to distance is known, this integral can be evaluated.

If the law is not known, but the force can be measured at each point,

then the work done can still be found, for $\int_0^s f\,ds$ is the area beneath a graph of force against distance bounded by the ordinates at 0 and s. The work done is thus represented by the area shaded in Fig. 2.4.

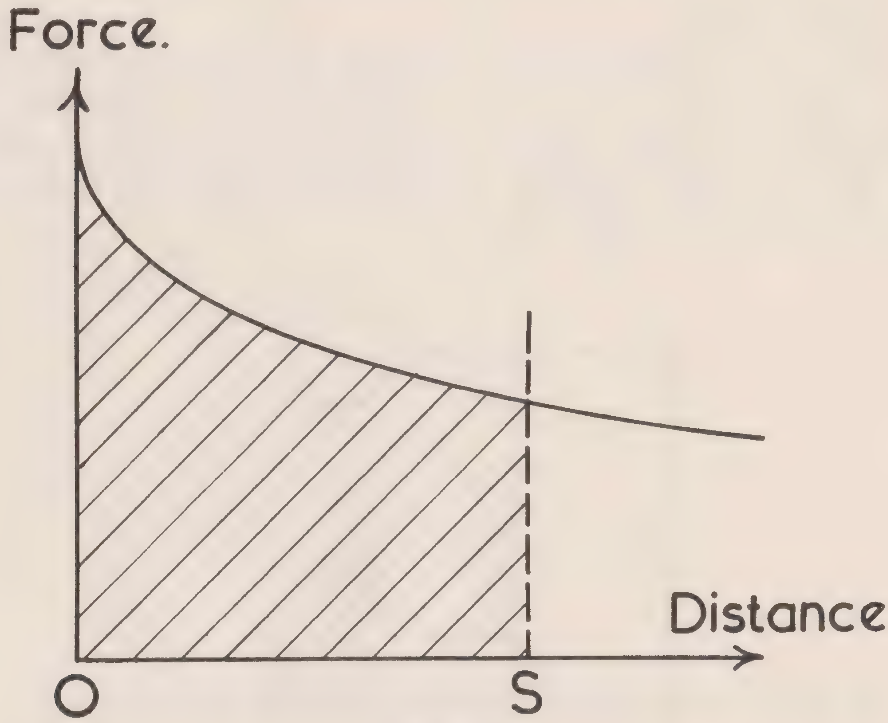


Fig. 2.4

Example 5. *A horse is towing a barge along a canal, the towrope making an angle of* 8° *with the towpath. If the tension in the rope is* 120 *lb, how much work does the horse do in* $\frac{1}{4}$ *mile?*

Tension in rope, i.e. force = 120 lb-wt.

Distance moved by barge = 1320 ft

Distance moved by barge in direction of force

$$= \{1320 \cos 8°\}\text{ft}$$

$$\text{Work done} = \{120 \times 1320 \cos 8°\}\text{ft.lb-wt.}$$

$$= 156{,}816 \text{ ft.lb-wt.}$$

2.11 Energy

In order to lift a body from one position to another it is necessary to do work against the force of gravity on the body. If the body is then allowed to return to its starting position it can be made to perform the same amount of useful work. It is as though work can be stored up in the body to be released when needed. This stored-up work is called the *Energy* of the body; it is measured by the amount of work necessary to get the body to its elevated position.

Work can be stored in many ways; for example, a moving body can be made to do work as it comes to rest; a compressed spring or gas does

work as it expands, and a cork under water will do work as it bobs to the surface.

In this branch of physics we shall be concerned mainly with two varieties of energy: that possessed by a body as a result of its position, called *Potential Energy*, and that possessed as a result of its motion, called *Kinetic Energy*.

Since energy is merely stored-up work it will have the same dimensions as work, i.e. $[ML^2T^{-2}]$, and will be measured in the same units—ergs in the c.g.s. system. It is also a scalar quantity, and all varieties of energy can be added together provided that they are expressed in appropriate units (see page 30).

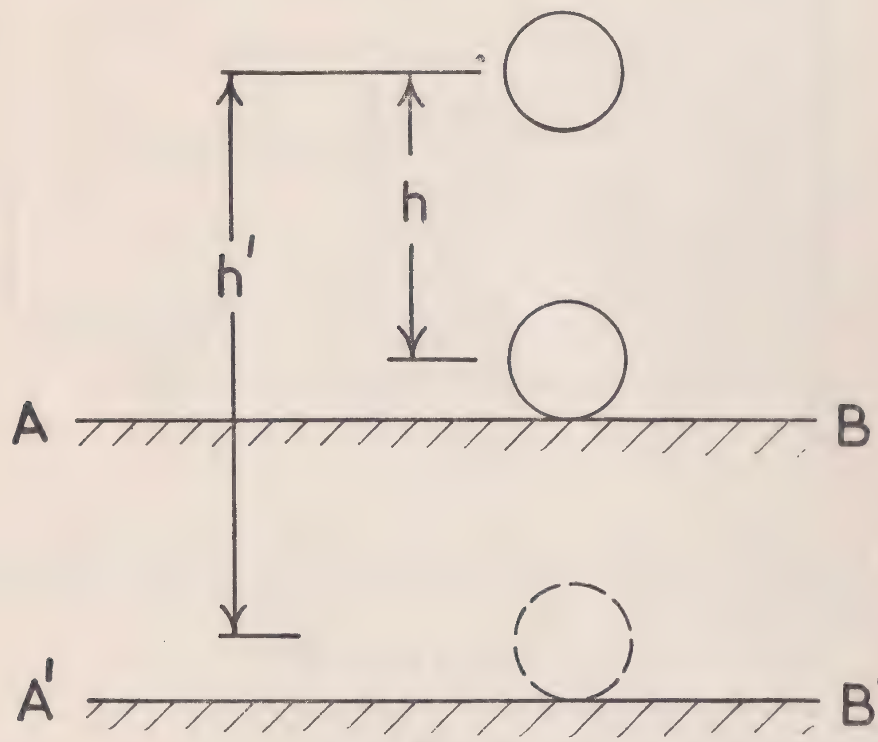


Fig. 2.5

Potential Energy

The potential energy of a body is the work stored in a body when it is elevated above a plane on which it would normally stand, or the work done against gravity when the body is lifted from the plane. If the body is of mass m gm, then the downward force on it is mg dynes and the work needed to lift it through a distance h cm as in Fig. 2.5 is mgh ergs. The potential energy of the body in the upper position is thus mgh ergs. Notice that this is the potential energy relative to the plane AB; if alternatively the body had been lifted from $A'B'$ then it would have possessed a different value of potential energy, namely mgh'. This is always true of potential energy, it only has a value *relative to a given starting-point*. In any problem it is usually possible to choose the starting-point that is most convenient for the given situation.

Kinetic Energy

Kinetic energy is the energy stored in a body when in motion, and is equal to the work done in getting it into that state of motion from rest.

If a body of mass m is accelerated uniformly from rest to a velocity v in a distance s, its acceleration is given by:

$$v^2 = 2as$$

and the force accelerating it must be

$$\begin{aligned} f &= ma \\ &= \frac{mv^2}{2s}. \end{aligned}$$

This force moves the body through a distance s whilst accelerating up to the velocity v, and hence, by the time it has reached this velocity, it will have done an amount of work given by the product (force $\times$ distance).

$$\begin{aligned} \text{Therefore work done} &= f \times s \\ &= \frac{mv^2}{2s} \cdot s \\ &= \tfrac{1}{2} mv^2. \end{aligned}$$

If now the application of this force stops, the body will continue to move with the same velocity v; no further work will be done upon it, and hence, so long as it maintains this velocity, its kinetic energy will be given by:

$$\text{Kinetic Energy} = \tfrac{1}{2} mv^2 \qquad . \quad . \quad . \quad (7)$$

2.12 Conservation of Energy

Two forms of energy have been discussed in the section above, but in the last 200 years many other physical quantities have been recognised as forms of energy; the more common ones are now called heat, chemical energy, electrical energy and radiant energy (such as light).

The relation between these quantities was first clearly recognised in 1798 by Rumford when boring cannon. He found that the heat evolved by the boring process, when performed by a blunt tool operating under water, was always roughly proportional to the work done in rotating the boring machine. The first thorough investigation of the question was made by Joule, working in the years 1840–50. He measured the amount of heat produced by the expenditure of both electrical and mechanical work, and found that the expenditure of a given amount of work always produced the same amount of heat. This suggested that heat is a form of energy and that the work had not been wasted but converted into another form of energy, or that heat and mechanical work are interchangeable at a *fixed* rate.

In Joule's day heat and mechanical work were measured in different

units. He found that 1 British Thermal Unit of heat appeared for the expenditure of 772 ft.lb-wt. of work, but we now regard 778 as a more accurate figure.

If heat and mechanical work are both forms of the same physical quantity, energy, it should be possible to express either quantity in either unit, i.e. mechanical work can be expressed in heat units provided that the number of the units is divided by 778 and vice versa.

If such is the case, then the mechanical work used in any experiment is numerically equal to the heat produced—or the total energy content of the system is unchanged.

These ideas have been extended to all other forms of energy leading to the *Principle of Conservation of Energy* which states that the *total energy in any closed system is a constant* although this energy may be transformed into many varieties.

In practice it is not always evident that energy is conserved; in a machine, for example, very rarely is the useful work done by the machine as large as the energy consumed by it, and hence it appears that energy has been lost. If, however, careful account is taken of the energy that appears as heat in the bearings, of the energy carried away from the machine as sound waves, of the kinetic energy of the air swept round with the flywheel and of all the other inconspicuous ways in which energy can leave the system, then it is found that in fact energy is rigorously conserved.

Example 6. *A mass of* 12 *lb hangs on a string* 13 *ft long. It is pulled aside to a distance of* 5 *ft from the vertical and then released. Find the velocity with which the mass passes through its lowest position.*

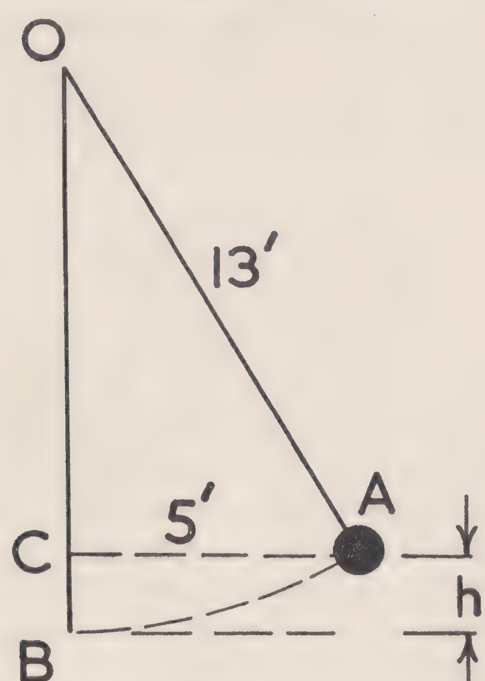


Fig. 2.6

From the diagram:

$$OC = \sqrt{13^2 - 5^2}\ \text{ft}$$
$$= 12\ \text{ft}$$
$$\text{But } OB = 13\ \text{ft}$$
$$\text{Thus } CB = h = 1\ \text{ft}$$

Potential energy of bob at A relative to horizontal plane through B

$$= 12 \times 32{\cdot}2 \times 1\ \text{ft.poundals.}$$

Kinetic energy of bob at A = zero.

Total energy at A = 386·4 ft.poundals.

If the bob has a velocity v ft.sec^{-1} when it returns to B:

Potential energy of bob at B = zero.

Kinetic energy of bob at $B = \frac{1}{2} . 12 . v^2$

$= 6v^2$ ft.poundals.

Total energy of bob at $B = 6v^2$ ft.poundals.

If energy is conserved, this gives:

$$6v^2 = 386{\cdot}4$$
$$\text{or } v \simeq 8\ \text{ft.sec}^{-1}.$$

2.13 Power

The *rate* at which a machine performs work is defined as the *Power* of the machine.

$$\text{Thus [Power]} = \left[\frac{\text{Work}}{\text{Time}}\right]$$
$$= \left[\frac{ML^2T^{-2}}{T}\right]$$
$$= [ML^2T^{-3}].$$

The c.g.s. unit of power is thus the gm.cm.^{2}sec^{-3} which is also called the erg per second.

This is a very small unit, and so in practice a unit of 10^7 ergs per sec, or 1 joule per second is used; this is renamed the *watt*. The M.K.S. unit of power is the unit of work divided by the unit of time, i.e. the joule per second, and is thus also the watt.

In the f.p.s. system of units the unit of power is the ft.poundal per second; the gravitational unit is the ft.lb-wt. per sec, but a larger unit is also in use, 550 ft.lb-wt per second being called 1 *Horse-Power.* 1 hp is equal to 746 watts.

Some of the units of energy or work in use at present have been evolved 'backwards' out of the unit of power; for example 1 joule per sec is 1 watt, thus 1 joule is 1 watt.second.

The watt.second is a unit of energy. It is rather a small unit, and so a larger multiple, the *Kilowatt-hour* (equal to 3,600,000 joules), is commonly used in the electrical industries.

Example 7. *A motor developing* 40 *hp is required to drive a boat at* 15 *mph. What would be the tension in a tow-rope if the boat were being towed at the same speed?*

In 1 second the boat travels 22 feet. Let the water resistance to its motion be R lb-wt, then work done in 1 sec $= 22\ R$ ft.lb-wt and rate of working $= 22\ R$ ft.lb-wt per sec.

$$\text{But rate of working of engine} = 40 \text{ hp}$$
$$= 40 \times 550 \text{ ft.lb-wt per sec.}$$
$$\text{Thus } 22\ R = 40 \times 550$$
$$\text{or } R = \frac{40 \times 550}{22} \text{ lb-wt}$$
$$= 1000 \text{ lb-wt}$$

If the boat were being towed at constant speed, the tow-rope would have to apply a tension equal to this resistance.

Thus tension in tow-rope = 1000 lb-wt.

SUMMARY OF NEW QUANTITIES INTRODUCED IN THIS CHAPTER

Quantity	*Dimensions*	*c.g.s. unit and Derived units*	*f.p.s. unit and Derived units*	*M.K.S. unit*	*Gravitational unit*
Momentum	MLT^{-1}	gm.cm.sec^{-1} or dyne.sec	lb.ft.sec^{-1}	kg.m.sec^{-1}	—
Force .	MLT^{-2}	dyne	poundal	newton	gm-wt = 981 dynes lb-wt = 32·2 poundals
Impulse	MLT^{-1}	dyne.sec	poundal.sec	newton.sec	gm-wt.sec lb-wt.sec
Work .	ML^2T^{-2}	erg joule = 10^7 ergs	ft.poundal	joule	cm.gm-wt ft.lb-wt
Energy .	ML^2T^{-2}	erg joule = 10^7 ergs kilowatt-hour = 36×10^5 joules	ft.poundal	joule	cm.gm-wt ft.lb-wt
Power .	ML^2T^{-3}	erg.sec^{-1} joule.sec^{-1} = 1 watt	ft.poundal.sec^{-1}	watt	ft.lb-wt.sec^{-1} 1 hp = 550 ft.lb-wt.sec^{-1}

EXERCISES 2

1. State Newton's second law of motion and use it to obtain a relation between force, mass and acceleration.

   A simple pendulum is hung inside the cabin of an aircraft flying at a constant height. It is observed to be deflected through an angle θ from the vertical when the aircraft increases speed with a constant acceleration a. Give a diagram showing the forces acting on the bob,

indicate on it which way the aircraft is travelling, and explain why the pendulum bob is deflected. Deduce an expression for $\frac{a}{g}$.

If $\theta = 5$, find the time taken to increase speed from 150 to 250 m.p.h. (Cambridge G.C.E. Advanced level.)

2. An observer in a closed railway carriage is equipped with a delicate spring balance and a plumb-line. While the carriage is at rest he determines the weight of a piece of metal and the position of the freely hanging plumb-line. What information would he derive from simultaneous observations with these two pieces of equipment when the carriage is in motion?

What would be observed inside the carriage if it was (*a*) going up an incline of 1 in 10 with uniform speed, (*b*) moving up the same incline with an acceleration of 3·2 ft.sec.$^{-2}$?

(Cambridge Univ. Schol., King's College Group.)

3. Given that 1 ounce = 28·3 grams and that 1 inch = 2·54 centimetres, g = 32·2 ft./sec.^{2}, determine:
   (i) the number of gms. wt. in 1 poundal,
   (ii) the number of kilowatts in 1 horse-power.
   (1 watt = 10^7 ergs/sec.) (London Univ. Inter. B.Sc.)

4. A train approaching a station travels two successive quarters of a mile in 24 and 30 seconds respectively. If the retardation is uniform, find the retarding force in lb. wt. per ton of the train.

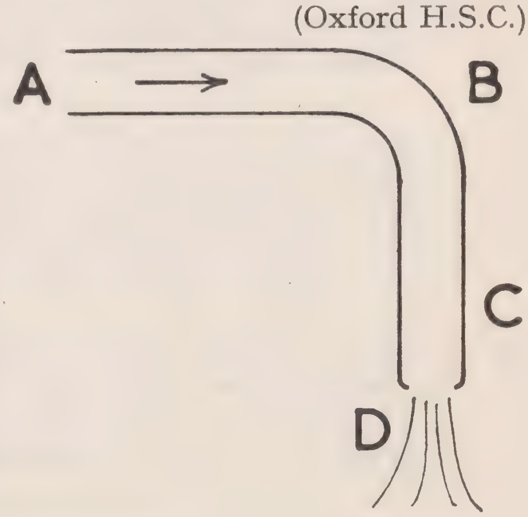
If the retardation is then changed so that the train comes to rest in the next quarter-mile, find the new retarding force, assumed constant.

(London Univ. Inter. B.Sc.)

5. State the law of conservation of momentum, and Newton's experimental law of impact. Describe how you would demonstrate ONE of these principles experimentally.

A uniform fine chain, of mass m gm. per centimetre length, is suspended so that its lower end just touches a horizontal table, and is then allowed to fall. Show that, at the instant when a length x cm. has reached the table, the force exerted on the table is 3 mgx dynes. (Oxford H.S.C.)

6. A stream of water flows with a velocity of 10 cm. per sec. along a tube ABC, BC being at right angles to AB, as in the diagram, and issues from a nozzle D with a velocity of 30 cm. per sec. Draw a diagram showing the velocities of the stream at A and at D, and the change in velocity that has occurred; determine this change in magnitude and direction.



The area of cross-section of ABC is 0·5 sq. cm. What is the change of momentum of the water per second, and what force is required to produce this change? (Cambridge G.C.E. Advanced level (Part).)

7. A seagull weighs 2 lb. and has an effective wing area of 1 ft^2. If the densityof air is 0·06 $lb.ft^{-3}$, calculate what vertical air speed is required for the bird to hover.
(Cambridge Univ. Schol., King's College Group (Part).)

8. The products of combustion of a rocket are ejected from it with a velocity v. If a fraction k of the initial weight of the rocket consists of fuel, find the velocity which the rocket will attain when all fuel is burnt. (Manchester Univ. Schol.)

9. Raindrops reaching the earth from clouds at 2000 ft. have a velocity of 20 m.p.h. Compare the kinetic energy of a drop with its change of potential energy and make any relevant observations.
(Cambridge Univ. Schol., King's College Group.)

10. A man on a bridge finds there a stone of mass 1 oz. at a height 30 ft. above the water below. He throws it a further 20 ft. into the air, and allows it to fall into the water. Calculate the kinetic energy of the stone on entering the water. Trace the origin of this energy as far as you can, and discuss what becomes of it. (Oxford Univ. Schol.)

11. What do you understand by the *conservation of energy*? Illustrate your answer by reference to the energy changes occurring (*a*) in a body whilst falling to and on reaching the ground, (*b*) in an X-ray tube.

The constant force resisting the motion of a car, of mass 1,500 kgm., is equal to one-fifteenth of its weight. If, when travelling at 48 km. per hour, the car is brought to rest in a distance of 50 metres by applying the brakes, find the additional retarding force due to the brakes (assumed constant) and the heat developed in the brake-drums.
(Northern Univ. H.S.C.)

12. A steel ball is dropped from a height of 200 cm. and, after rebounding from the floor, rises to a height of 150 cm. Calculate an upper limit for the rise in temperature of the ball during the impact. In which other ways may energy have been dissipated in the process?
(Specific heat of steel = 0·11 cals./gramme.)
(Oxford Univ. Schol.)

13. A mass of 5 cwt. falls freely from a height of 12 ft. on to an inelastic pile whose mass is 15 cwt. If the pile is driven in 9 in. at each blow, find the average resistance of the ground in tons weight. Find also the fraction of the kinetic energy that is lost (i.e., not used to overcome the resistance) at each blow. (London Univ. Inter. B.Sc.)

14. What is meant by (*a*) the principle of the conservation of momentum, (*b*) the principle of the conservation of energy?

The velocity of a rifle-bullet can be found by firing the bullet into a box of sand which is suitably suspended. Sketch the apparatus, describe the experimental procedure and show how the desired result

is derived from the observations. Indicate what happens to the energy originally present in the moving bullet.

(Northern Univ. G.C.E. Advanced level.)

15. What is understood by (*a*) the principle of the *conservation of energy*, (*b*) the principle of the *conservation of momentum*?

A bullet of mass 20 gm., travelling horizontally at 100 metres per sec., imbeds itself in the centre of a block of wood, of mass 1 kgm., which is suspended by light vertical strings 1 metre in length. Calculate the maximum inclination of the strings to the vertical.

(Northern Univ. H.S.C.)

16. State Newton's laws of motion, and describe how you would attempt to test them experimentally.

A bullet, of mass 50 gm., travelling at 600 metres per sec., strikes a wooden block, of mass 20 kg., which is suspended from long vertical threads so that it is free to swing. The bullet penetrates the block completely, and emerges on the other side travelling at 400 metres per sec. in its original direction. Calculate the vertical height through which the block rises. (Oxford G.C.E. Advanced level.)

17. State and explain what is meant by the law of conservatism of momentum and describe an experiment to illustrate it.

A wooden pendulum bob is moving east with a velocity of 2 m./sec. and is hit by a bullet travelling north-west with a velocity of 100 m./sec. If the mass of the pendulum bob is 1 kg. and that of the bullet 100 gm., and if the bullet remains embedded in the bob, find the velocity after impact. (Cambridge H.S.C.)

18. The total mass of a train is 600 tons. Find the greatest horse-power the engine can develop on the level if the greatest speed attainable on a horizontal track is 50 m.p.h. and the resistances to motion are 10 lb. wt. per ton.

Find also the maximum speed, v m.p.h., attainable up a slope of 1 vertically to 80 along the slope, if the horse-power then developed is the maximum horse-power on the level multiplied by $\left(1 - \frac{v}{20}\right)$.

(London Univ. Inter. B.Sc.)

19. A lorry, of mass 2 tons, with the engine shut off, runs down a slope of 1 in 50 with an acceleration of $0{\cdot}2$ ft./sec.^{2}. Calculate the resistances to motion in lb. wt.

Assuming the resistances to remain unaltered, find what horse-power is required to take the lorry up the same slope at a constant speed of 25 miles per hour. (London Univ. Inter. B.Sc.)

20. A train of weight 250 tons travels at 60 m.p.h. on a level track, the resistance being 100 lb. per ton. Calculate the h.p. of the engine. If a rear coach of 25 tons weight is slipped, find the time that must elapse before the train will attain a speed of 65 m.p.h. Assume that the resistance is independent of velocity in the range 60 to 65 m.p.h.

(Manchester Univ. Schol.)

21. Define work and power. Show that if a particle moves against a resistance then it cannot exceed a certain speed depending upon the power exerted to maintain its motion.

A train consists of three engines which work at rates H_1, H_2 and H_3 when the train is travelling at full speed against a frictional resistance R per unit weight. Show that the tensions in the two couplings are equal if:

$$W_2 (H_1 + H_3) = H_2 (W_1 + W_3),$$

where W_1, W_2 and W_3 are the weights of the three engines.

(Cambridge Univ. Schol., Girton and Newnham Colleges.)

22. A man of weight 60 kg. climbs a mountain 1000 m. high and loses 500 gm. weight as perspiration. Assuming that his temperature remains constant, and that the human body is 20% efficient on converting chemical energy into mechanical work, how much heat does his body lose by means other than evaporation of perspiration?

(Latent heat of evaporation of water at body temperature 600 cal.gm.$^{-1}$.) (Cambridge Univ. Schol., King's College Group (Part).)

CHAPTER 3

MACHINES, FRICTION AND IMPACT

3.1 Introduction

In the previous two chapters the basic concepts used in mechanics have been discussed; this chapter shows how these ideas can be applied to the performance of a real machine. Some of the apparent discrepancies in the theory of a machine are accounted for by the action of friction and an explanation of these forces is given later in the chapter; finally, the principles of conservation of energy and momentum are applied to the theory of the motion of two bodies on impact.

3.2 Machines

In many cases energy is released in one part of an apparatus and has to be used to do work on another part of the system; for example, in a car, energy is released in the cylinder of the engine by burning petrol and is used to turn the road wheels. Any mechanical device which helps to apply energy in the performance of useful mechanical work is called a *Machine*. In practice machines can become very complex, but they are all made by interconnecting a number of simple mechanisms, some of which are mentioned below.

In any machine, work is put in by an *Effort* which is, of course, a force. Work done by the effort is the product (force $\times$ distance moved by point of application of force parallel to its direction). If this effort and distance are F_1 and d_1 respectively, then:

$$\text{work put into the machine} = F_1 d_1.$$

If the machine moves a load F_2 through another distance d_2, then:

$$\text{work done on the load} = F_2 d_2.$$

If energy is to be conserved, this leads to:

$$F_1 d_1 = F_2 d_2$$

$$\text{or } \frac{d_1}{d_2} = \frac{F_2}{F_1} \qquad (1)$$

The ratio d_1/d_2, i.e. $\dfrac{\text{distance moved by effort}}{\text{distance moved by load}}$, is called the *Velocity Ratio* of the machine, while the ratio $\dfrac{F_2}{F_1}$, i.e. $\dfrac{\text{load}}{\text{effort}}$, is called the *Mechanical Advantage*, so that in a perfect machine the velocity ratio is equal to the mechanical advantage.

Both the velocity ratio and mechanical advantage are ratios of

similar quantities. They are therefore dimensionless and are represented by pure numbers.

In all practical machines some energy is inevitably lost in overcoming friction, and some work is used in moving parts of the machine itself, so that the useful work done on the load is not equal to the work supplied by the effort; this means that the ratio $\frac{F_2 d_2}{F_1 d_1}$ is not usually equal to unity. The ratio $\frac{\text{Useful work done on load}}{\text{Total work done by effort}}$ is called the *Efficiency* of the machine.

$$\text{Note that } \frac{F_2 d_2}{F_1 d_1} = \frac{F_2}{F_1} \div \frac{d_1}{d_2}.$$

$$\text{Thus Efficiency} = \frac{\text{Mechanical Advantage}}{\text{Velocity Ratio}} \quad . \quad . \quad (2)$$

The efficiency of a machine is the ratio of two pure numbers and is thus a pure number itself and without dimensions.

The velocity ratio can always be found by considering the geometry of the machine, but the mechanical advantage can only be found by experiment, since it is not known how much energy is going to be lost in any particular machine. In fact this energy loss is very rarely a constant, but varies with the load the machine is carrying.

Simple Machines

The lever, consisting of a strong rod acting over a fulcrum, is probably the simplest form of machine. It can be used to exert large forces.

The lever need not be straight; a device such as a bell crank, used to change the direction of a force, can be treated as a lever.

Frictional effects are rarely important in the case of the lever, but the effort sometimes has to lift part of the weight of the lever itself and this lowers the efficiency of the system.

Winches may be considered as simple machines and treated as an extension of the bell crank.

Single pulleys may be used to change the direction of an effort and multi-sheaved pulleys to obtain a fairly high velocity ratio. Friction in a pulley system is however always fairly large and so the highest velocity ratios are obtained with differential pulley systems.

The inclined plane may be used as a machine either in the form of a simple wedge or else as a screw and nut. Very high velocity ratios may be obtained but the frictional losses are large, due to the sliding surfaces in contact. The efficiency of such a system is thus not very high.

Finally, simple mechanisms may include gear trains and belt or chain drives. These can all be made relatively free from friction and so of high efficiency.

3.3 Friction

One of the main reasons for the efficiency of a machine falling below a hundred per cent. is the presence of *friction* in the bearings. Friction is the name given to the force which opposes the motion of one body sliding over another.

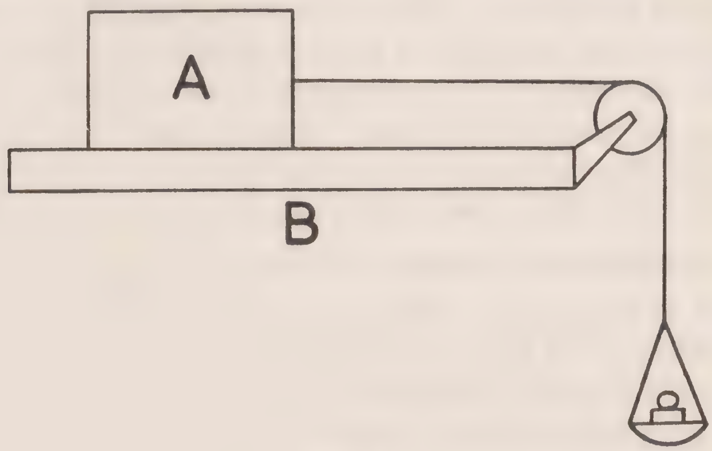


Fig. 3.1

The simple laws of friction, enunciated originally by Amontons in 1699, can be studied with the apparatus shown in Fig. 3.1. The block A is pulled over the flat surface B by a cord attached to a scale pan; initially A is at rest and the scale pan carries no load. Some weights are now added gently to the pan, but it is found that motion does not immediately ensue; this must mean that some other force is opposing the motion of A. This other force is of course the frictional force between the two surfaces, and if the block is to remain at rest then the frictional force must adjust itself always to be equal to the tension in the string but acting in the opposite direction; thus the frictional force and the tension in the string grow together as the load in the pan is increased.

As the loading of the pan continues, a point is reached where motion of the block begins. This means that there is now a resultant force acting on the block in the direction of the tension in the string, and indicates that the frictional force has reached a maximum value. Thus any further increase in the tension in the string is not counterbalanced by an increase in the frictional force.

This maximum value of the frictional force is called *Limiting Friction*.

Experiments could now be performed to find the effect on the value of limiting friction of varying (*a*) the nature of the surfaces in contact, (*b*) the size of the surfaces in contact and (*c*) the force pressing the two surfaces together. These experiments would show us that, for a given pair of surfaces in contact, the value of the limiting friction is proportional to the force pressing the two surfaces together, but is independent of the area of the surface of the block in contact with the plane.

If F_L is the limiting value of friction and R the force pressing the

surfaces together (R is usually called the 'normal reaction') we can write:

$$F_L \propto R$$

$$\text{or } F_L = \mu R \qquad \text{(3)}$$

where μ is a constant for the pair of surfaces in contact.

μ is called the *Coefficient of Static Friction*, since it is found from the value of F occurring just before the block slips, i.e. while it is static. To distinguish it from other coefficients to be defined later, it will be written as μ_S. F_L cannot, of course, be measured directly, but we can measure the tension in the cord, and this is equal to the frictional force so long as the block is at rest, hence μ_S can be found experimentally.

Rearranging Equation (3) then gives $\mu_S = F_L/R$

Both F_L and R are forces, thus μ_S is the ratio of two similar quantities; the coefficient of friction is therefore dimensionless and represented by a pure number. Another interesting fact emerges from these experiments on a sliding block; motion of the block should begin when the tension in the string exceeds the value of limiting friction by a very small amount, the resultant force on the block is then very small and consequently its acceleration should also be small. In practice, however, much larger accelerations than would be expected are observed; this suggests that the frictional force drops to a value rather lower than its limiting value as soon as the block starts moving. The tension in the string, of course, stays the same and hence the resultant force on the block becomes larger and so increases its acceleration.

The value of the frictional force when the block is moving is called the *Dynamic Friction* (*sliding* friction or *kinematic* friction) and is written as F_D; this is also found to be proportional to the force pressing the two surfaces together, hence

$$F_D = \mu_D R \qquad \text{(4)}$$

where μ_D is the *Dynamic Coefficient of Friction.*

F_D can be found by varying the load on the scale pan until the block, having been started with a gentle push, continues to move at a constant speed, i.e. with no acceleration. The resultant force on the block is then zero and the tension in the cord must be equal to F_D.

Limiting or static friction is sometimes called stiction; its value is of great importance to engineers who design small mechanisms which must start moving smoothly and without jerks.

3.4 Theories of Friction

The experimental laws of friction contain one fact which at first appears very surprising—the frictional force is independent of the area of the surfaces in contact—and it is only recently that an adequate theory has been offered for this.

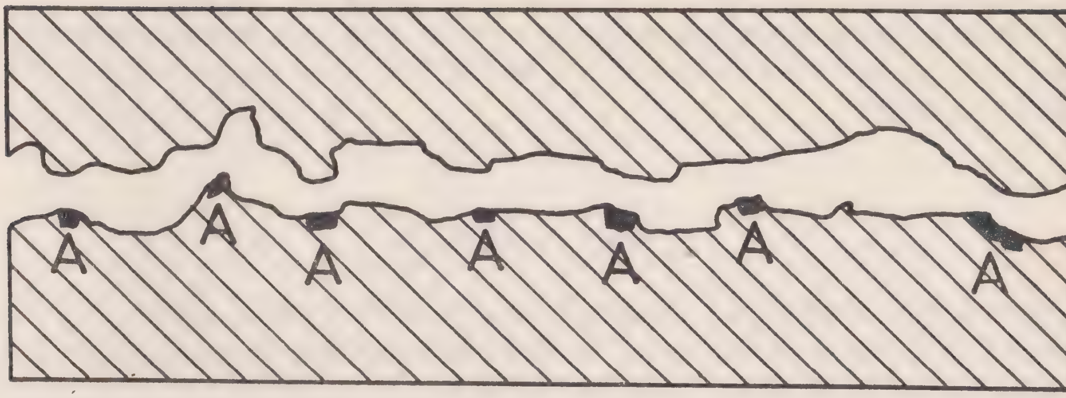


Fig. 3.2

When two surfaces are placed together, however flat they may be, they actually touch each other in a very few points indeed. A highly magnified view of the surfaces just before they are placed together would appear rather like Fig. 3.2. When they are in contact, the weight of the upper plate will be supported on a few points such as those marked A. The area of these points is very small, in fact recent experiments have shown that it is of the order of one-millionth part of the total area of the surface, consequently the pressure on these points, equal to the load divided by the area supporting it, will be very large indeed. To illustrate this we may think of a disc of copper, diameter 3 cm and mass 10 gm (roughly the same as a penny), lying on another copper plate. The area of the points in contact will be roughly $\pi \times 1{\cdot}5^2 \times 10^{-6}$ cm^2, and pressure on these points will be approximately $\dfrac{10}{\pi \times 1{\cdot}5^2 \times 10^{-6}}$ gm-wt.cm^2 or 10 tons per square inch.

At pressures as high as this, metal deforms in rather the same way as putty, plastic flow takes place (see Chapter 8) and the surfaces are squashed together as in Fig. 3.3. As this squashing takes place, the area in contact increases and so the pressure decreases, until it reaches the lowest value that can produce plastic flow, whereupon the process stops.

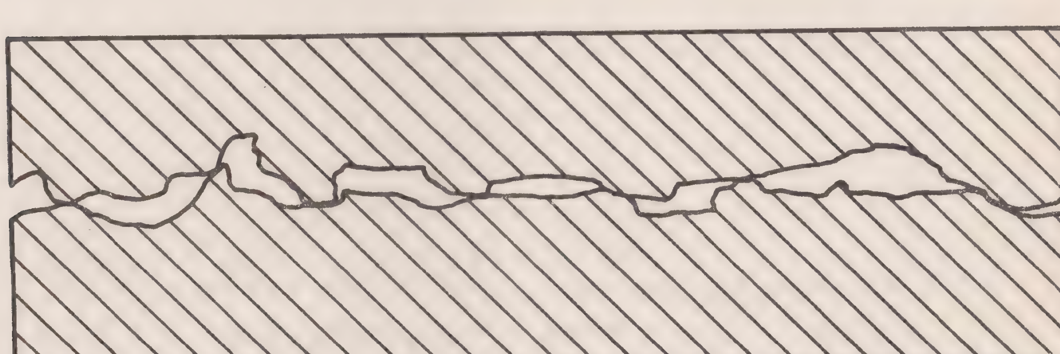


Fig. 3.3

For any particular metal, the minimum pressure for plastic flow is a constant, P say, thus if A is the area of the plates in true contact after they have been squashed together, and R the force pressing the plates together, then:

$$P = R/A$$

$$\text{or } A = \frac{1}{P}.R$$

Remembering that P is a constant, it is seen that the contact area between the plates is proportional to the force pressing them together but is not related in any way to the total area of the plates.

When regions of the plates are forced into intimate contact as described above, the two layers of molecules brought together will adhere just as they do when built up in a solid metal; if we then try to slide one surface over the other, the numerous small necks of metal which have welded the surfaces together will have to be sheared off. The force needed to do this is proportional to the contact area (see Chapter 8), but this contact area is proportional to the load, thus the frictional force should be proportional to the load—a fact which is borne out by experiment.

When slipping takes place, experiments have shown that the movement is really a series of jerks. The block, initially at rest, starts to move when the tension in the string overcomes the limiting friction. It moves on a small distance, so reducing the tension in the string, which in turn allows the block to come to rest, whereupon the process is repeated. The average tension in the string is thus rather lower than that needed to overcome limiting friction, and the dynamic coefficient of friction is rather smaller than the static one.

Example 1. *A block of mass* 150 *gm stands on a horizontal plane. A cord from the block passes over a pulley at the edge of the plane, and is then attached to a scale pan. It is found that the block just starts to move when the total load in the pan is* 105 *gm. If the dynamic coefficient of friction is* 0·8 *times the static coefficient, find how far the block would slide on the plane if projected with a velocity of* 300 *cm.sec*$^{-1}$.

$$\text{Limiting friction force} = 105g \text{ dynes.}$$

$$\text{Normal reaction to plane} = 150g \text{ dynes,}$$

$$\text{thus } \mu_S = \frac{105g}{150g}$$

$$= 0{\cdot}7,$$

$$\text{hence } \mu_D = 0{\cdot}7 \times 0{\cdot}8 = 0{\cdot}56.$$

$$\text{But } F_D = \mu_D \,.\, R.$$

$$\text{Or sliding frictional force} = 0{\cdot}56 \times 150 \times 981 \text{ dynes.}$$

$$\text{Now } f = ma,$$

$$\text{thus deceleration of the block} = \frac{0{\cdot}56 \times 150 \times 981}{150} \text{ cm.sec}^{-2},$$

and substituting numerical values in $v^2 = u^2 + 2as$ gives

$$0 = (300)^2 - 2 \times 0{\cdot}56 \times 981 \times s$$

$$\text{or } s = \frac{90000}{2 \times 0{\cdot}56 \times 981} \text{ cm}$$

$$= 82 \text{ cm.}$$

3.5 Laboratory Methods of Measuring the Coefficient of Friction

One simple method using a slide and scale pan has already been described; an alternative method involves the use of an inclined plane.

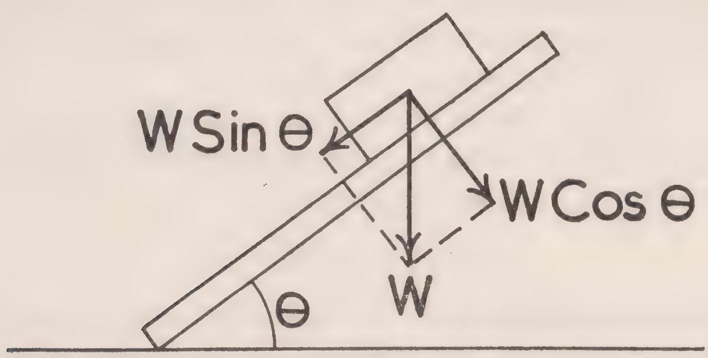


Fig. 3.4

In Fig. 3.4 the force tending to slide the block down the plane is $W \sin \theta$ and the normal reaction is $W \cos \theta$; as the inclination is increased the sliding force becomes larger until it exceeds limiting friction and the block starts to move; if this happens at θ_L, which is called the

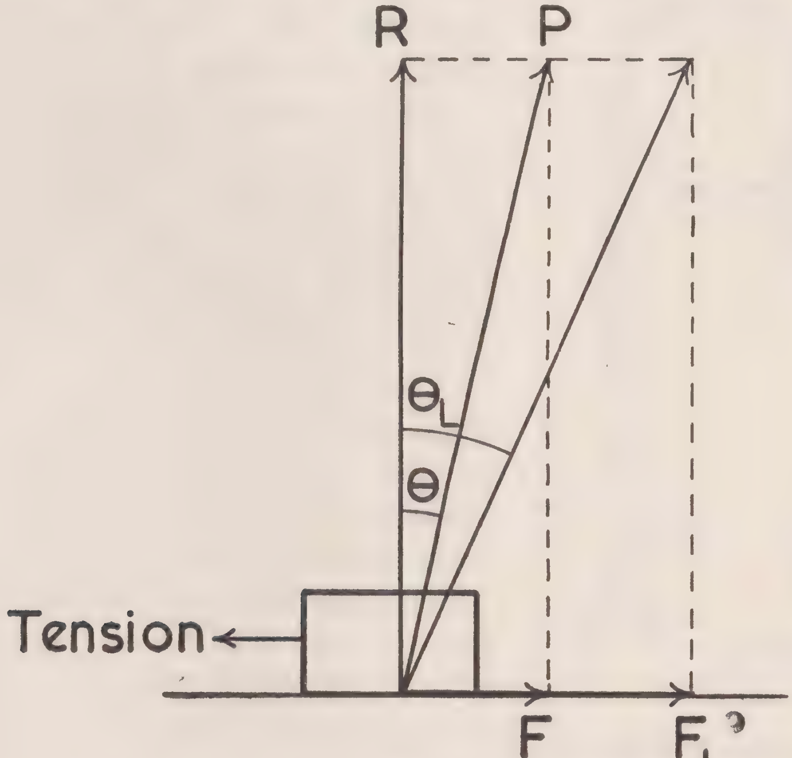


Fig. 3.5

limiting slope or *Angle of Repose* (since it is the steepest slope on which the block will remain at rest), then the coefficient of static friction is given by:

$$\mu_S = \frac{W \sin \theta_L}{W \cos \theta_L}$$

$$= \tan \theta_L \; . \qquad (5)$$

We can consider this from another point of view. If an attempt is made to move a block along a plane, then the forces exerted by the plane *on the block* are the frictional force F (Fig. 3.5) and the reaction R. These can be combined into the resultant P which makes an angle θ with R. This angle is called the *Angle of Friction.*

As the force urging the block to move increases, the frictional force increases to its limiting value F_L and the angle of friction also increases to a limiting value θ_L. From the diagram:

$$\tan \theta_L = F_L/R = \mu_S \; . \qquad (6)$$

which is the same as Equation (5) developed for the angle of repose. Thus the *limiting* value of the angle of friction is equal to the angle of repose.

3.6 Work done against Friction

The force of friction always opposes the motion of a body, hence if a body is moved along a distance d against a frictional force F, an amount of work equal to $F \times d$ will be done; this fact is often used to measure the rate of working of a machine.

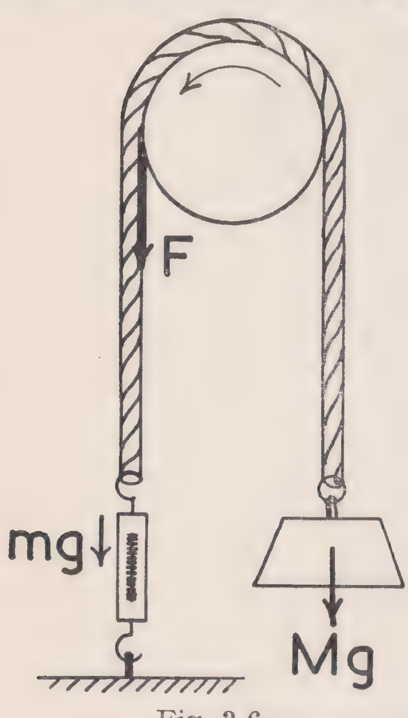


Fig. 3.6

A belt, or band-brake, is stretched around a pulley of the machine as shown in Fig. 3.6, so that when the pulley rotates the frictional force between it and the belt supports a mass M. In practice small speed variations would cause the position of the weight to be very unsteady, therefore a spring balance is included at the other end of the belt so that any tendency for the weight to fall will be opposed by increased tension in the spring of the balance and vice versa.

If the speed of the wheel is adjusted so that the load is neither rising nor falling, then its weight is completely supported by the frictional force exerted on the belt by the pulley plus the tension of the spring balance. If this frictional force is F and the reading of the spring balance is m, then:

$$F + mg = Mg$$

$$\text{or } F = (M - m)g \qquad . \quad . \quad . \quad (7)$$

Now the belt must apply an equal and opposite frictional force to the rim of the pulley; if this is of radius R, then, during one revolution, a point on the rim moves a distance $2\pi R$ against the force F, and hence does an amount of work equal to $2\pi RF$. Thus we have:

$$\text{Work done per revolution} = 2\pi RF$$

$$= 2\pi Rg(M - m)$$

and if the pulley makes N revolutions per second, then the work done per second, or rate of working is given by:

$$\text{Power} = 2\pi RNg(M - m) \qquad . \quad . \quad (8)$$

A device such as this for measuring the power of a machine is called a *Dynamometer*; a good example is to be found in a method of measuring Joule's mechanical equivalent of heat. See *Experimental Physics*, by C. B. Daish, M.Sc. and D. H. Fender, Ph.D., published by English Universities Press, Ltd.

3.7 Impact

If two bodies move towards each other and eventually collide, an *impact* occurs; if this is a head-on collision between two bodies moving along the line joining their centres of gravity (Fig. 3.7), it is called a *Direct Impact*, any other collision is called an *Oblique Impact*. See Fig. 3.8. The manipulation of problems involving impacts is treated very fully in books on Applied Mathematics. Here only simple problems sufficient to illustrate the physics of the process will be discussed.

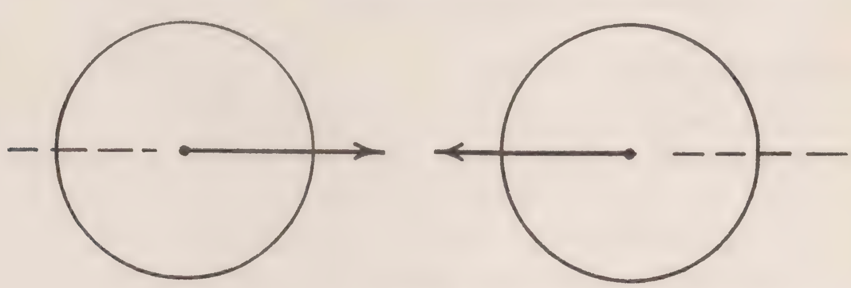


Fig. 3.7

The theory given below takes account only of the linear motion of the bodies and assumes that no rotation takes place; the bodies are thus treated as spheres so that on direct impact the resultant force passes through the centre of gravity and hence produces no rotation, also they are assumed to be perfectly smooth so that on oblique impact there is no tangential frictional force between the spheres to set them spinning. It is almost impossible to meet this last condition in practice.

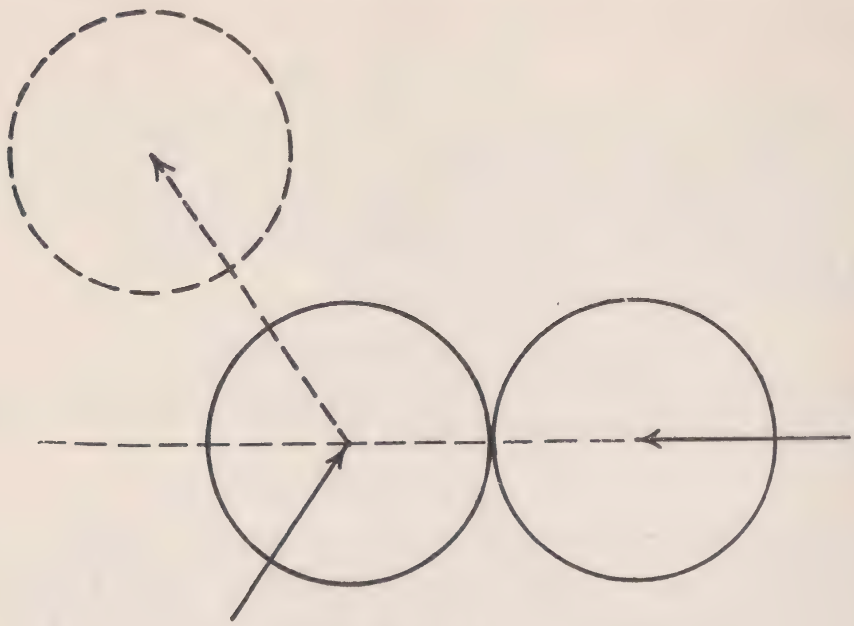


Fig. 3.8

Conservation of Momentum during Impact

During the actual collision between two bodies it is very difficult to state the nature or the magnitude of the forces acting between them. It is known, however, from Newton's third law, that the forces which the two bodies exert on each other are at all times equal but opposite in direction, and must of necessity act for the same time, i.e. for the interval during which the two bodies are in contact. Each body therefore receives the same magnitude impulse and hence suffers the same change in momentum, but in opposite directions. Remembering that momentum is a vector quantity, it is seen that the principle of conservation of momentum is satisfied during any impact.

Effect of Elasticity on Impact

The forces between two spheres when they collide may be very great, and in addition will be distributed over the very small area of contact between them. The deformation of the spheres is thus likely to be large.

In a later chapter the work done in deforming a body is calculated; so long as the body is perfectly elastic this work is stored in the body just as energy is stored in a compressed spring, and is all given up as the body regains its original shape. If, however, the body is imperfectly elastic, some of the work done on it produces a permanent deformation and when the body is allowed to relax, rather less energy comes out than was put in. At the extreme of this scale is the perfectly plastic body; all the work done on such a body produces permanent deformation and no energy can be recovered.

Similarly three cases of impact can be distinguished. If two perfectly elastic bodies collide, then some of the kinetic energy of the moving

bodies is used to deform them. This deformation increases until they are at rest relative to each other and the kinetic energy apparently lost is stored up in the bodies. In regaining their shape, the bodies force each other away so that they acquire a velocity relative to each other and the energy stored in the deformed bodies is all converted back into kinetic energy.

If, on the other hand, two plastic bodies collide, they deform until they are moving with a common velocity as one body. The kinetic energy lost is used to deform the bodies and is not recoverable.

In between these two cases we have the collision between imperfectly elastic bodies. As before, some kinetic energy is used to deform the bodies as they come to rest relative to each other, but on separation the kinetic energy gained is smaller than that lost on collision due to the imperfect elasticity of the spheres.

3.8 Direct Impact of Perfectly Elastic Spheres

Let the spheres be of mass m_1, m_2 (Fig. 3.9) and move with velocities u_1, u_2 respectively. (Velocities measured to the right are called positive.) If a collision is to occur, u_1 must be greater than u_2. After impact let the spheres move with velocities v_1, v_2 respectively, so that $v_1 < v_2$ if separation is to occur.

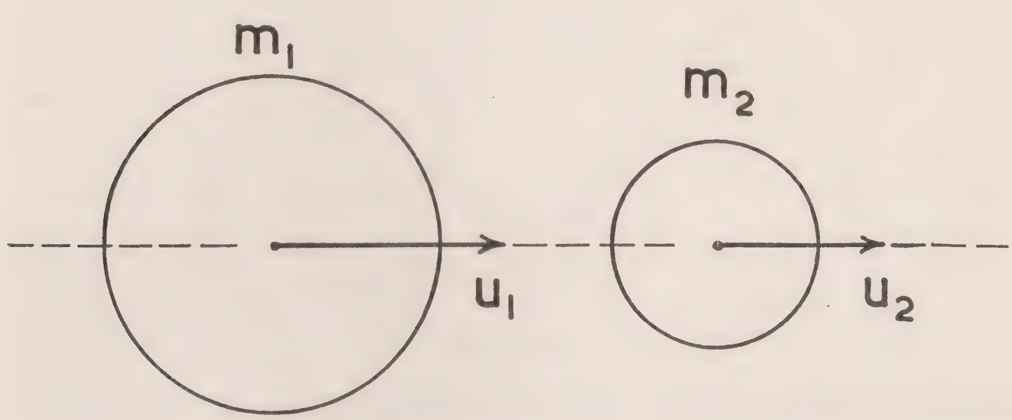


Fig. 3.9

Then from the principle of conservation of momentum:

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \qquad (9)$$

and from the principle of conservation of energy:

$$\tfrac{1}{2}m_1u_1^2 + \tfrac{1}{2}m_2u_2^2 = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 \qquad (10)$$

Equation (9) can be rewritten as:

$$m_1(u_1 - v_1) = m_2(v_2 - u_2) \qquad (11)$$

and Equation (10) as:

$$m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2) \qquad (12)$$

dividing each side of Equation (12) by the corresponding side of Equation (11) gives:

$$u_1 + v_1 = v_2 + u_2$$

$$\text{or } (u_1 - u_2) = (v_2 - v_1) \quad . \quad . \quad . \quad (13)$$

Now $(u_1 - u_2)$ is the relative velocity of approach of the two bodies and $(v_2 - v_1)$ is the relative velocity of separation; thus when two perfectly elastic bodies collide, their relative velocities before and after impact are the same.

3.9 Direct Impact of Two Plastic Bodies

With the notation of the previous section, imagine two spheres colliding and adhering to each other; thereafter let them move with a

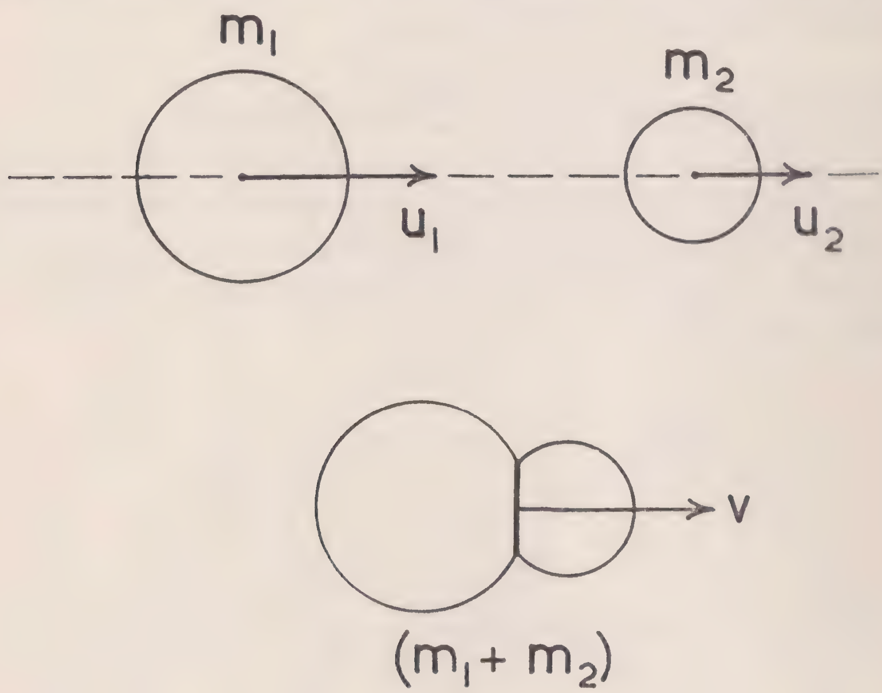


Fig. 3.10

combined velocity v (Fig. 3.10). From the principle of conservation of momentum we have:

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v$$

$$\text{or } v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2} \quad . \quad . \quad . \quad (14)$$

The energy expended in deforming the bodies can now be calculated, for:

$$\text{Energy before impact} = \tfrac{1}{2}m_1u_1^2 + \tfrac{1}{2}m_2u_2^2$$

$$\text{Energy after impact} = \tfrac{1}{2}(m_1 + m_2)v^2$$

Energy used in deforming the bodies

$$= \tfrac{1}{2}\{m_1u_1^2 + m_2u_2^2 - (m_1 + m_2)v^2\}$$

or substituting for v from Equation (14):

$$\text{Energy lost} = \tfrac{1}{2}\left\{ m_1u_1^2 + m_2u_2^2 - \frac{(m_1u_1 + m_2u_2)^2}{m_1 + m_2} \right\}$$

$$= \tfrac{1}{2} \cdot \frac{m_1m_2}{m_1 + m_2} (u_1 - u_2)^2 \qquad . \qquad . \qquad (15)$$

3.10 Direct Impact of Two Imperfectly Elastic Bodies

This problem is best considered by splitting the impulse into two phases. The first is from the instant when the bodies are just about to touch and the conditions are as in Fig. 3.11 (*a*) to the instant when each body has undergone maximum deformation and they are momentarily at rest relative to each other as in Fig. 3.11 (*b*). The second phase then starts and continues until the bodies separate again, when the conditions are as shown in Fig. 3.11 (*c*).

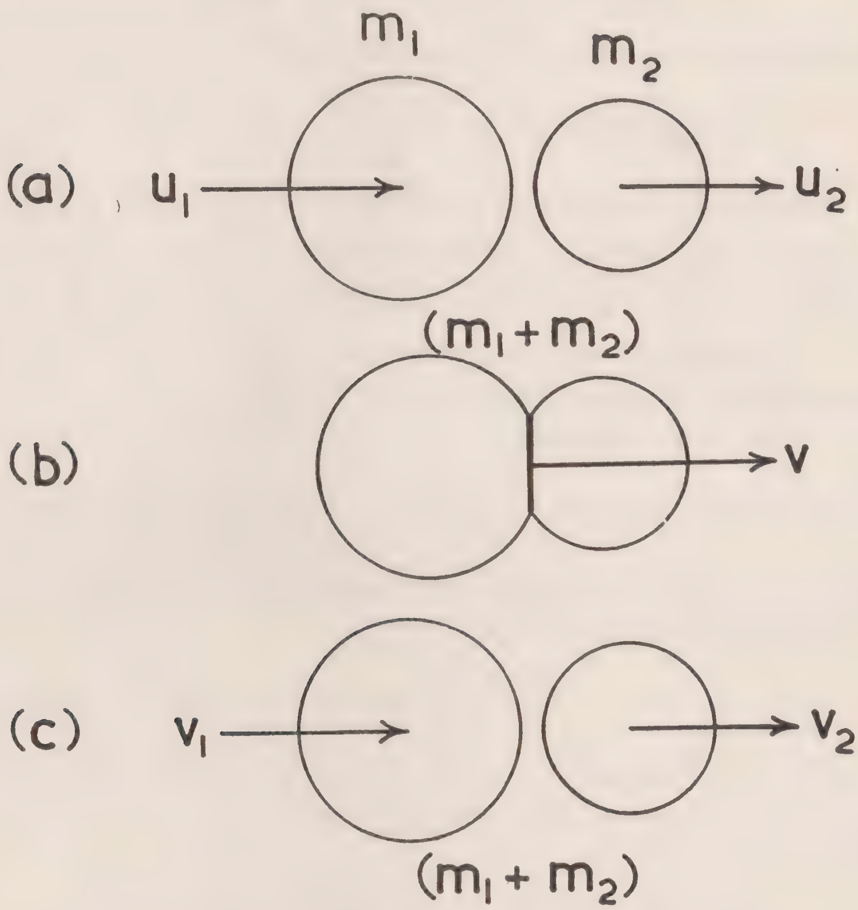


Fig. 3.11

The first phase of this collision follows exactly the conditions of the previous section and hence, as in Equation (14), we have

$$v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2}$$

and following Equation (15):

$$\text{Energy used to deform the bodies} = \frac{1}{2} \cdot \frac{m_1 m_2}{m_1 + m_2} (u_1 - u_2)^2 \quad . \quad (16)$$

A theoretical discussion of the second phase of the impact cannot be given since it is not known how much of the energy stored during deformation will be restored as the bodies regain their original shape; but if the set of final conditions as shown in Fig. 3.11 (*c*) can be found by experiment, then the proportion of energy restored can be calculated as follows.

From the principle of conservation of momentum:

$$(m_1 + m_2)v = m_1 v_1 + m_2 v_2 \quad . \quad . \quad . \quad (17)$$

also kinetic energy at beginning of second phase $= \frac{1}{2}(m_1 + m_2)v^2$
and kinetic energy at end of second phase $= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$,

Energy regained during second phase

$$= \tfrac{1}{2}\{m_1 v_1^2 + m_2 v_2^2 - (m_1 + m_2)v^2\} \quad . \quad . \quad (18)$$

and substituting for v from Equation (17) gives:

$$\text{Energy regained} = \frac{1}{2} \cdot \frac{m_1 m_2}{m_1 + m_2} (v_1 - v_2)^2 \quad . \quad . \quad (19)$$

Dividing corresponding sides of Equations (19) and (16) gives:

$$\frac{\text{Energy regained as bodies separate}}{\text{Energy stored as bodies are deformed}} = \left\{\frac{v_1 - v_2}{u_1 - u_2}\right\}^2 \quad . \quad (20)$$

Now $(u_1 - u_2)$ and $(v_2 - v_1)$ are the relative velocities of the two bodies when approaching and separating respectively.

Newton found by experiment that, for impacts which produce no serious deformation, the ratio $\frac{\text{velocity of separation}}{\text{velocity of approach}}$ can be regarded as a constant; this ratio is called the *Coefficient of Restitution*, and is designated by the symbol e, thus

$$\frac{v_1 - v_2}{u_1 - u_2} = e \quad . \quad . \quad . \quad . \quad (21)$$

The coefficient of restitution is the ratio of two velocities, thus it is dimensionless and represented by a pure number.

From Equation (19) we see that:

$$\frac{\text{Energy regained as bodies separate}}{\text{Energy stored as bodies are deformed}} = e^2 \quad . \quad . \quad (22)$$

Further, from Equation (19) we have:

$$\begin{aligned}\text{Energy regained} &= \frac{1}{2} \cdot \frac{m_1 m_2}{m_1 + m_2} (v_1 - v_2)^2 \\ &= \frac{1}{2} \cdot \frac{m_1 m_2}{m_1 + m_2} e^2 (u_1 - u_2)^2\end{aligned}$$

and subtracting this from Equation (16) gives the energy converted into heat, etc., during the collision and so 'lost' as kinetic energy.

$$\text{Thus kinetic energy lost} = \tfrac{1}{2} \cdot \frac{m_1 m_2}{m_1 + m_2}(u_1 - u_2)^2(1 - e^2) \qquad (23)$$

3.11 Methods of Measuring the Coefficient of Restitution

A convenient method of measuring the coefficient of restitution of a material available both as a flat plate and as a sphere is to measure the height of rebound of the sphere when dropped onto the plate. If the sphere is dropped from a height h_1 and rebounds to a height h_2, it is readily seen that

$$e = \sqrt{\frac{h_2}{h_1}}. \qquad (24)$$

The coefficient of restitution can also be calculated from the time that elapses between the release of the ball and the instant when it finally ceases to bounce.

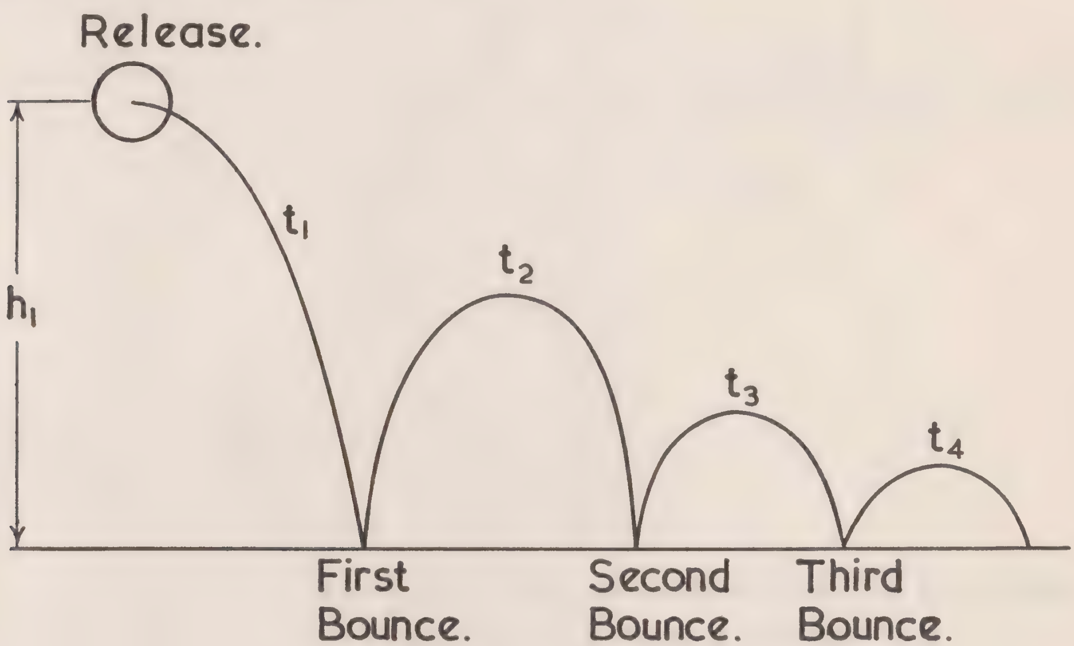


Fig. 3.12

Referring to Fig. 3.12, we have:

Time from release to first bounce is given by

$$h_1 = \tfrac{1}{2} g t_1^2$$

$$\text{or } t_1 = \sqrt{\frac{2h_1}{g}}.$$

Velocity on first impact $= \sqrt{2gh_1}$.

Velocity of rebound $= e\sqrt{2gh_1}$.

Hence time from first bounce to second bounce is given by substituting the appropriate values in the equation $s = ut + \frac{1}{2}at^2$,

$$\text{i.e. } 0 = e\sqrt{2gh_1}\,.\,t_2 - \tfrac{1}{2}gt_2^2$$

$$\text{or } t_2 = \frac{e\sqrt{2gh_1}}{g/2}$$

$$= 2e\sqrt{\frac{2h_1}{g}},$$

$$\text{similarly, } t_3 = 2e^2\sqrt{\frac{2h_1}{g}}$$

$$t_4 = 2e^3\sqrt{\frac{2h_1}{g}}, \text{ and so on.}$$

Now total time of bouncing is given by

$$T = t_1 + t_2 + t_3 \ldots \text{ to infinity}$$

$$= \sqrt{\frac{2h_1}{g}}\,(1 + 2e + 2e^2 + 2e^3 \ldots \text{ to infinity}).$$

But e is always smaller than unity, hence the series $2e + 2e^2 + 2e^3 \ldots$ is convergent and the sum to infinity is $2e/(1 - e)$.

$$\text{Thus } T = \sqrt{\frac{2h_1}{g}}\left(1 + \frac{2e}{1 - e}\right)$$

$$= \sqrt{\frac{2h_1}{g}}\left(\frac{1 + e}{1 - e}\right)$$

$$\text{or } e = \frac{\sqrt{\dfrac{gT^2}{2h_1}} - 1}{\sqrt{\dfrac{gT^2}{2h_1}} + 1} \quad . \quad . \quad . \quad . \quad (25)$$

Summary of Units Introduced in this Chapter

All of the quantities derived in this chapter, i.e. Velocity Ratio, Mechanical Advantage, Efficiency of a Machine, Coefficient of Friction and Coefficient of Restitution, are dimensionless quantities and so are pure numbers.

EXERCISES 3

1. A man hauls on a rope (not necessarily vertical) which goes over a frictionless pulley to a weight W resting on the ground immediately below the pulley. Discuss the factors determining the maximum value of W which the man can lift.

   What do you consider to be the essential features of a simple machine? Illustrate your answer by reference to some idealised machines. (Cambridge Univ. Schol., King's College Group.)

2. A body moves from rest with uniform acceleration. Find expressions for its velocity and the distance it has travelled after time t. Describe how you would verify ONE of these expressions.

A small body slides down a plane inclined to the horizontal at an angle of 45°, and then continues sliding on a horizontal surface. If the coefficient of friction between the body and the surfaces is $\sqrt{2} - 1$, show that the body is brought to rest in a distance equal to the length of the inclined plane.

(Cambridge Univ. Schol., Girton and Newnham Colleges.)

3. Define (i) *coefficient of friction*, (ii) *angle of friction*.

A body of 5 lb. wt. is just prevented from sliding down a rough inclined plane by a force of 2 lb. wt. acting up a line of greatest slope. When this force is increased to 3 lb. wt. the body just begins to slide up the plane. Prove that the coefficient of friction between the body and the plane is $\sqrt{3}/15$. (London Univ. Inter. B.Sc.)

4. State the laws governing the frictional force between a body and the surface over which it is sliding. Describe simple experiments by which the validity of these laws could be investigated. Distinguish between the coefficient of static and sliding friction.

A loaded toboggan, of mass 160 lb., is set into motion on a horizontal ice surface by a man who exerts a constant horizontal force of 10 lb. weight for 3·0 sec. If the appropriate coefficient of sliding friction is 0·020, calculate the distance travelled by the toboggan after the man lets go. (Northern Univ. H.S.C.)

5. State the meaning of *coefficient of friction* and describe a method of measuring it for iron on wood.

A body rests on a plane which may be inclined at any angle, θ, to the horizontal. The coefficient of friction between the plane and body surfaces is $1/\sqrt{3}$. If the angle θ is gradually increased from zero, at what value of θ will the body begin to slide down the plane, and what will be its acceleration when $\theta = 60°$?

(London Univ. G.C.E. Advanced level.)

6. State the laws of sliding friction.

A block rests on a rough plane inclined at an angle of 30° to the horizontal. The horizontal force needed to drag the block directly up the plane is fifteen times the horizontal force needed to drag it directly down. Find the coefficient of friction between the block and the plane.

(Oxford Univ. Schol.)

7. State the laws of friction, and discuss how far they are obeyed in practice.

A conveyor belt of coefficient of friction μ is used to lift parcels from ground-level to a platform at height h. What is the greatest angle the belt can make with the horizontal so that a parcel put on with the velocity of the belt, u, just reaches the platform?

(Oxford Univ. Schol.)

8. A rope is coiled a few times round a cylindrical post and it is found that a small pull (T_1) applied at one end is able to sustain a much

larger pull (T_2) at the other end. Show that $T_2/T_1 = e^{\mu\theta}$, where μ is the coefficient of friction between rope and post and θ the angle of lap in radians of the rope around the post. Suggest an experimental method for verifying the formula and show how you would plot your results and use the graph to determine the value of μ.

A band brake makes contact over half the circumference of a pulley of 6 inches diameter which revolves at the rate of 200 revolutions per minute. The tensions on the two sides are 300 lb. wt. and 100 lb. wt. respectively. Find (*a*) the coefficient of friction between the band and pulley, (*b*) the horse-power delivered to the pulley.

(Northern Univ. G.C.E. Schol. level.)

9. Define *coefficient of restitution*, and describe how it can be determined for a ball dropped on to a horizontal surface.

A small ball-bearing is dropped from a height of 90 cm. on to a horizontal floor, the coefficient of restitution being 0·9. Find the time intervals between (*a*) the first impact and the second impact, (*b*) the second impact and the third impact.

(Cambridge G.C.E. Advanced level.)

10. A staircase, consisting of ten steps, descends at an angle of 45°. A small india-rubber ball rolls off the top step, strikes the centre of the second and continues to bounce downwards. Find the number of bounces that occur before it reaches the bottom, assuming that the ball is perfectly smooth and elastic. (Manchester Univ. Schol.)

11. Define *coefficient of restitution* and describe how you would determine its value for two steel spheres. You may suppose that each sphere has a small hook attached to it.

A steel ball, mass 100 gm., drops from a height of 100 cm. on to a slate floor and rebounds for the first time to a height of 81 cm. Calculate (*a*) the total distance the ball travels until it comes to rest on the floor, (*b*) the total time (from the instant of release) which elapses until this occurs. How much energy does the ball lose as a result of the first three impacts with the floor?

(Northern Univ. G.C.E. Schol. level.)

12. Describe how you would measure the coefficient of restitution between two materials.

A ball is thrown on to a smooth horizontal floor and strikes it with a velocity of 5 metres per sec. at an angle of 60° to the horizontal. If the coefficient of restitution between ball and floor is 0·6, find the horizontal range and maximum height of the first rebound.

(Oxford G.C.E. Scholarship level.)

13. Describe how you would use the ballistic pendulum to measure the coefficient of restitution between two materials, and explain the theory of the experiment.

A ball A, of mass 200 gm., travelling at 50 cm. per sec., collides directly with another ball B of mass 300 gm., which is at rest. Find the velocity of each ball after the impact, if the coefficient of restitution between them is 0·6. (Oxford G.C.E. Advanced level.)

14. Describe the ballistic balance (or ballistic pendulum), and explain how you would use it to measure the coefficient of restitution between two substances.

A small sphere travelling at 50 cm.sec.$^{-1}$ strikes a smooth fixed plane at an angle of 40° to the normal. Find the velocity of the sphere after impact if the coefficient of restitution between sphere and plane is 0·3. (Oxford H.S.C.)

15. Give an account of the laws of friction, and describe how you would determine the coefficient of sliding friction between two surfaces.

A halfpenny, sliding along a rough horizontal board, collides obliquely, when its velocity is 30 cm. per sec., with a similar coin which is at rest, the direction of the motion being at 60° to the line of centres. If the coefficient of restitution between the two is 0·6, the coefficient of friction between each coin and the board is 0·1, and the coefficient of friction between the coins is negligible, find how far each travels after the collision before coming to rest.

(Oxford G.C.E. Scholarship level.)

CHAPTER 4

ROTATIONAL AND SIMPLE HARMONIC MOTION

4.1 Introduction

So far in this book we have investigated only the translational motion that occurs when a force is applied to a body; experience tells us, however, that generally a rotation is produced as well, and in this chapter these rotational effects will be studied.

It will be found that many of the equations for rotational motion are similar to the equations already developed for linear motion, this close resemblance is a big help to the memory.

4.2 Moment of a Force

If a force F acts at a point A on the lamina shown in Fig. 4.1, then in general, the action of the force will cause the lamina both to slide along and to twist around. The tendency of the force to produce a twist about any given point in the body is measured by the *Moment* of the force *about that point*, it is defined as the product of the force and the perpendicular distance from the line of action of the force to the point under consideration.

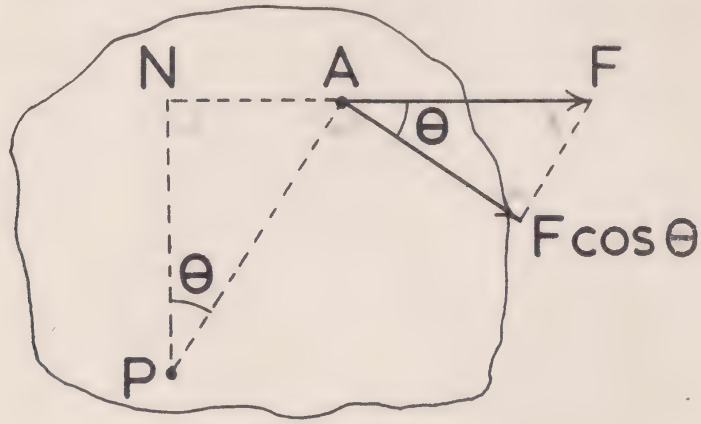


Fig. 4.1

The turning effect or moment of the force in Fig. 4.1 about the point P is given by:

$$\text{Turning effect} = F \times \text{distance } PN \qquad (1)$$
$$= F \times PA \cos\theta$$
$$= PA \times F \cos\theta \qquad (2)$$

from which it follows that the turning effect is also equal to the distance from the point of application of the force to the pivot P multiplied by the component of the force perpendicular to this line.

No symbol is regularly adopted for moment, but we shall use γ, therefore, from Equation (1)

$$\begin{aligned}[\gamma] &= [\text{Force} \times \text{distance}] \\ &= [MLT^{-2} \times L] \\ &= [ML^2T^{-2}]. \qquad (3)\end{aligned}$$

The c.g.s. unit of moment is thus the gm.cm^2.sec^{-2} and the f.p.s. unit is the lb.ft^2.sec^{-2}. These are often called the dyne.cm and the poundal.ft respectively, while the corresponding gravitational units are the gm-wt.cm and the lb-wt.ft. It will be noticed that these units are the same as the units of work although the quantity measured is quite a different one (as will be seen in more detail below); to avoid confusion, the c.g.s. unit of work has already been renamed the erg, while in the f.p.s. system, the gravitational unit of work is called the ft.lb-wt and the unit of moment the lb-wt.ft.

4.3 Moment as a Vector Quantity

Although work and moment have the same dimensions, both being the product of a force and a distance, they are really quite distinct quantities since different distances are involved in the two definitions.

Thus work = force × distance moved *parallel* to the force,

but moment = force × distance *perpendicular* to the force.

Now force and distance are both vectors, but the product of two vectors is a scalar quantity if they are parallel and a vector quantity if they are perpendicular; thus work is a scalar quantity, but moment is a vector quantity. This is perhaps the most important difference between them.

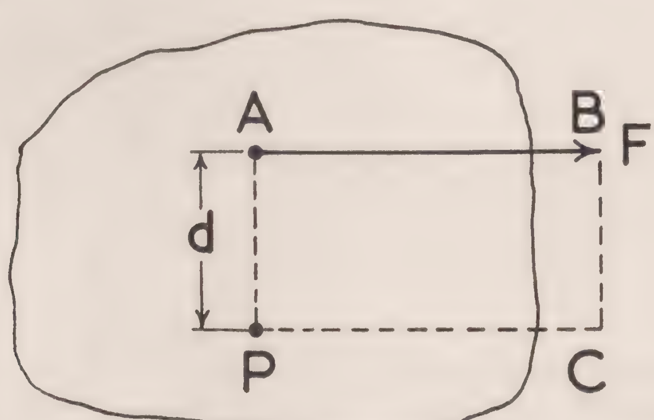


Fig. 4.2

The moment of the force F about the point P in Fig. 4.2 is given by:

$$\gamma = F \times d.$$

If the vector quantities F and d are represented on a scale drawing by the lengths AB, AP respectively, then the vector quantity γ, equal to

the product $F \times d$, is represented on the same drawing by the product $AB \times PA$, i.e. by the area $PABC$. Thus the *direction* of the vector quantity γ must be the *direction* of the plane $PABC$.

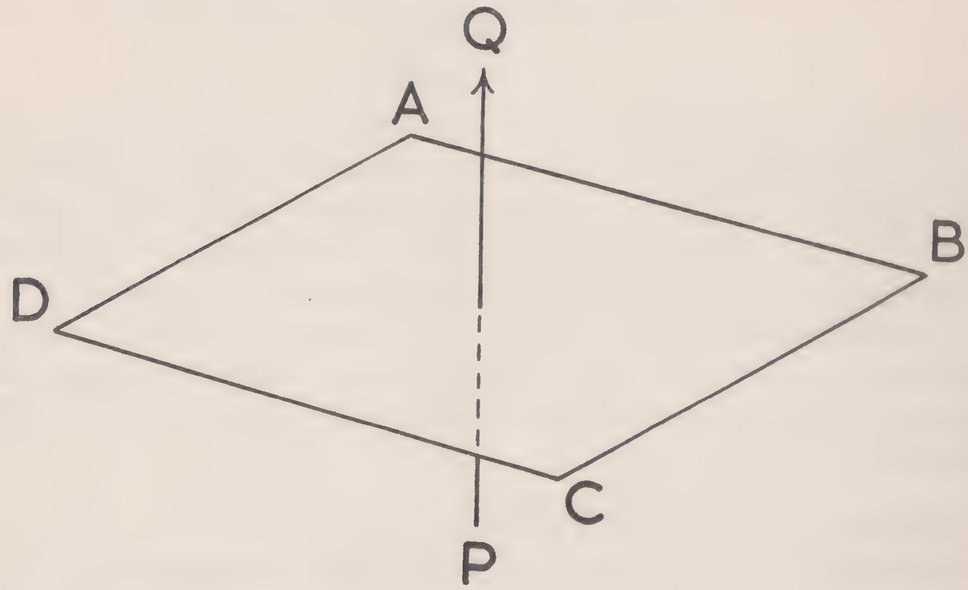


Fig. 4.3

Now the only way that we can describe the direction of a plane in space is to give the direction of a line at right angles to the plane; thus the direction of the plane $ABCD$ in Fig. 4.3 is fixed if we give the direction of the line PQ at right angles to it. Hence, referring again to Fig. 4.2, the direction of the moment would be perpendicular to the plane $PABC$, i.e. sticking in to the paper.

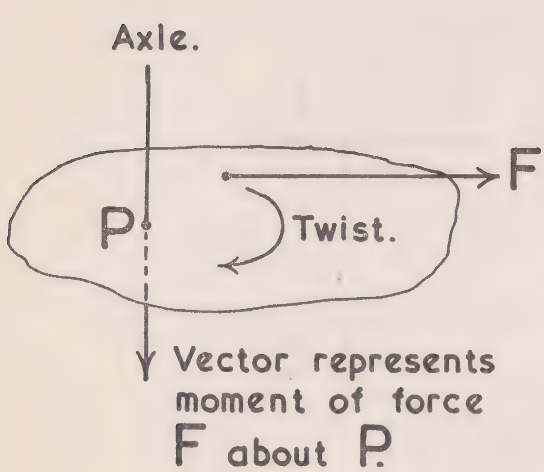


Fig. 4.4

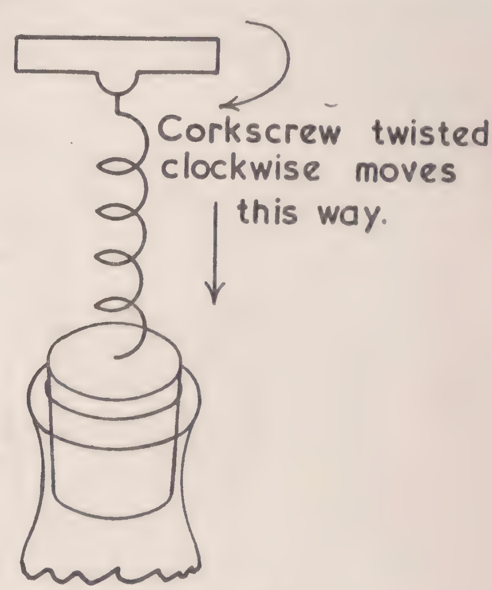


Fig. 4.5

If the force F were causing a turning motion about an axle fitted through the lamina at P, then the vector representing the moment of the force would point along the axle. To remove ambiguity of direction we adopt the right-handed corkscrew rule, i.e. the arrow on the vector

points in the same direction as a right-handed corkscrew would move if twisted in the same direction as the plane; this is illustrated in Figs. 4.4 and 4.5.

Addition of Moments

If a body is subjected to a number of forces all acting in the same plane, then the moments of each of these forces about any point in the plane will all be represented by vectors in the same direction, i.e. in Fig. 4.6 all the vectors will be perpendicular to the plane of the paper

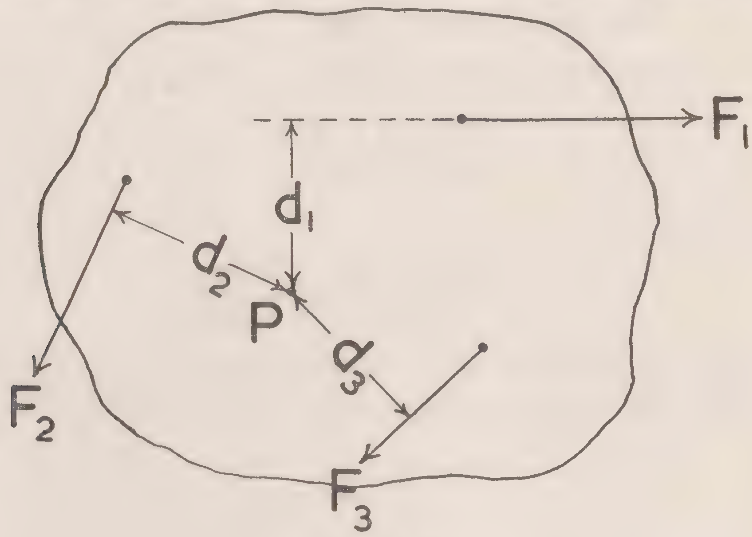


Fig. 4.6

through P. Now vectors in the same direction can be added by ordinary algebraic methods, thus to find the total moment applied to the lamina in Fig. 4.6, we merely add the moments due to the individual forces.

4.4 Centre of Gravity or Centroid

Any body can be considered to be made up of a number of small elements each of mass δm and having a weight $g\delta m$. All of these

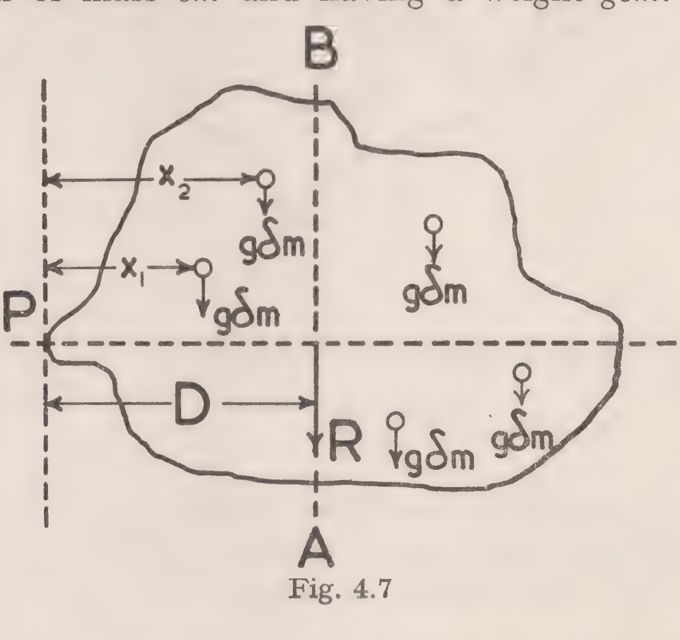


Fig. 4.7

weights will constitute a set of parallel forces. It is possible to find the line of action of the resultant force as follows.

For simplicity, consider a lamina as shown in Fig. 4.7; take moments about any convenient point P on the body, then the total moment about P is given by:

$$\gamma = g\delta m \,.\, x_1 + g\delta m \,.\, x_2 \ldots \text{etc.} \qquad (4)$$

If the resultant R acts at a distance D from P, then:

$$\gamma = R \,.\, D \qquad (5)$$

$$\text{But } R = (g\delta m + g\delta m \ldots \text{etc.}),$$

$$\text{hence } \gamma = (g\delta m + g\delta m \ldots \text{etc.})\, D.$$

$$\text{Thus } D = \frac{g\delta m x_1 + g\delta m x_2 \ldots \text{etc.}}{g\delta m + g\delta m \ldots \text{etc.}} \qquad (6)$$

which can be written as:

$$D = \frac{g\Sigma(\delta m \,.\, x_1)}{g\Sigma\delta m} = \frac{\Sigma(\delta m \,.\, x_1)}{\Sigma\delta m} \qquad (7)$$

where the sign Σ indicates that all the terms for each element into which the body has been divided have to be added together.

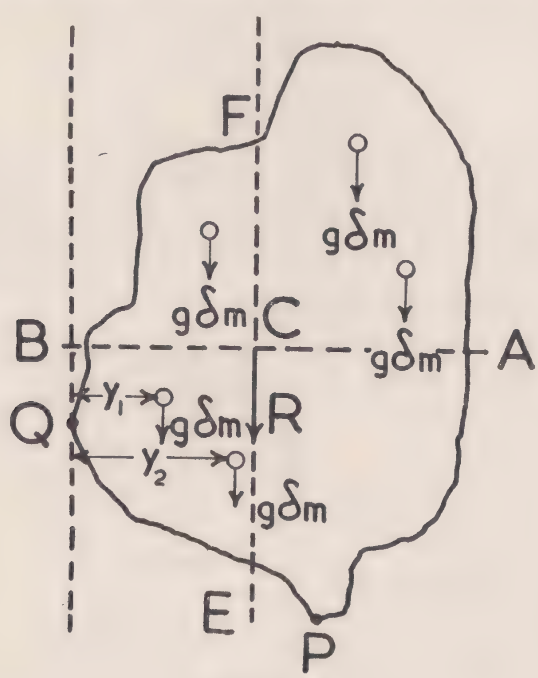


Fig. 4.8

Now $\Sigma\delta m$ is the total mass M of the body,

$$\text{thus } D = \frac{\Sigma(\delta m \,.\, x_1)}{M} \qquad (8)$$

In any practical case when the shape of the body is known, the body can be divided into suitable small elements and the value of $\Sigma(\delta m \,.\, x_1)$ can be found; thus the distance D, giving the distance from P of the

line of action of the resultant force, can be found. This is the line in which the weight of the body acts *when in this position,* and is drawn as AB in Fig. 4.7. It is fixed on the body and if the body is pivoted at any point on this line, the resultant moment is zero, hence the body experiences no tendency to twist under its own weight.

If the body of Fig. 4.7 is now turned through a right angle as shown in Fig. 4.8, the process outlined above can be repeated, taking moments this time about the point Q. This yields a new distance D' given by:

$$D' = \frac{\Sigma(\delta m \, . \, y_1)}{M} \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

giving the distance at which the resultant acts from Q when the body is in this new position. This line has been drawn in as EF on the diagram, and it is seen that the body, when in this position, will have no tendency to turn under its own weight if it is pivoted at any point on EF.

These two lines AB and EF intersect at C, thus if the body is pivoted at this point, it will show no tendency to turn under its own weight when set in either position or at any intermediate orientation.

It appears, then, that the moment of the weight of the body about C is always zero. This can happen only if the weight of the body acts at C, hence C is known as the *Centre of Gravity,* since the gravitational force on the body appears to act at this point.

Other names used are *Centroid* and *Centre of Mass*, for the weight of a body is merely the gravitational attraction exerted by the Earth on the body's mass; if this force acts at the point C, then the body is behaving as though the whole of its mass is concentrated at C; it must be noted however that there are some aspects of dynamics where this idea of concentration of mass cannot be used. For methods of calculating the position of the centroid the student is referred to books on Applied Mathematics.

4.5 The Chemical Balance

The balance that is used in the laboratory for 'weighing' really compares the moments produced by two weights, and so the theory of the balance may conveniently be considered here.

The balance consists of a beam (Fig. 4.9) supported by a centrally placed knife-edge K resting on a flat surface; the beam carries at its ends two more knife-edges on which rest flats supporting the pans of the balance. If the left- and right-hand pans carry masses M_1, M_2 respectively, and are themselves of mass P_1, P_2, while the corresponding arms of the beam are of length d_1, d_2, then the total moment about the knife-edge K is given by:

$$\gamma = (M_2 + P_2)gd_2 - (M_1 + P_1)gd_1 \quad . \quad . \qquad (10)$$

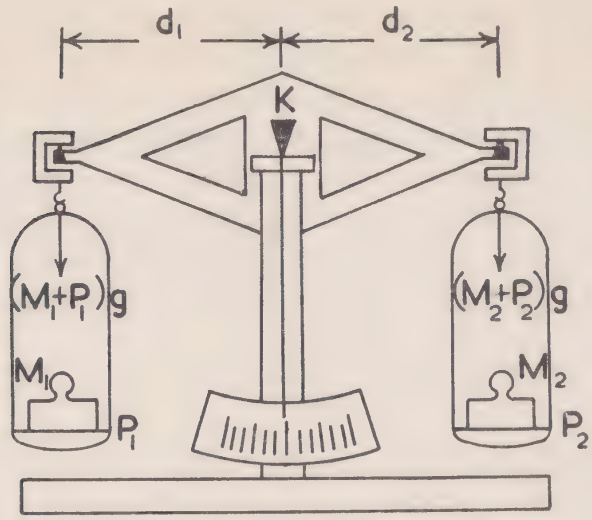


Fig. 4.9

If the masses M_1 and M_2 are adjusted until the balance beam shows no tendency to turn, then the total moment must be zero,

$$\text{thus } (M_2 + P_2)gd_2 = (M_1 + P_1)gd_1$$

$$\text{or } M_1d_1 + P_1d_1 = M_2d_2 + P_2d_2 \quad . \quad . \quad . \quad (11)$$

Now if the balance has previously been adjusted so that the beam does not turn when the pans are empty, then:

$$P_1d_1 = P_2d_2 \quad . \quad . \quad . \quad . \quad (12)$$

and substituting this in Equation (11) gives:

$$M_1d_1 = M_2d_2 \quad . \quad . \quad . \quad . \quad (13)$$

$$\text{or } \frac{M_1}{M_2} = \frac{d_2}{d_1} \quad . \quad . \quad . \quad . \quad (14)$$

Thus, if the arms of the balance are adjusted to be equal, M_1 is equal to M_2; the mass of an unknown body can thus be found by counter-poising it against a set of weights (i.e. calibrated masses). The accuracy of the weighing is limited only by the accuracy with which the arms can be made of equal length.

Double Weighing

Any slight inequality in the lengths of the arms of a balance will cause it to give an inaccurate result, but this can be corrected by a process known as double weighing.

The body to be weighed is first placed in the left-hand pan and weighed; it is then moved to the right-hand pan and the process repeated. If the arms are of different lengths, two incorrect answers will be obtained, but the correct weight can be derived from them as follows:

Assume that, apart from the unequal arms, the balance is otherwise

in correct adjustment; then if the body to be weighed has a true mass m, but needs a mass m_1 placed in the right-hand pan to counterpoise it, from Equation (14):

$$md_l = m_1 d_r \qquad . \quad . \quad . \quad . \quad (15)$$

where d_l, d_r are the lengths of the left and right arms respectively.

Now move the unknown mass into the right-hand pan, if a mass m_2 is needed in the left-hand pan to counterpoise it, then

$$m_2 d_l = md_r \qquad . \quad . \quad . \quad . \quad . \quad (16)$$

$$\text{Thus } m = \frac{m_1 d_r}{d_l} \text{ from (15).}$$

$$\text{But } \frac{d_r}{d_l} = \frac{m_2}{m} \text{ from (16)}$$

$$\text{hence } m = \frac{m_1 m_2}{m}$$

$$\text{or } m = \sqrt{m_1 m_2} \qquad . \quad . \quad . \quad . \quad (17)$$

from which the true mass of the body can be calculated.

Also, the ratio of the arms is given by:

$$\frac{d_l}{d_r} = \frac{m}{m_2}$$

$$= \frac{\sqrt{m_1 m_2}}{m_2}$$

$$= \sqrt{\frac{m_1}{m_2}} \qquad . \quad . \quad . \quad . \quad . \quad (18)$$

4.6 Factors in the Design of a Good Balance

Three conditions are of paramount importance when designing a balance: firstly the beam and knife-edges must be strong enough to carry the maximum load expected of the balance without bending or breaking; secondly, the smallest weight that is required to be measured must produce a noticeable deflection of the beam, in other words, the balance must have sufficient sensitivity; and finally, the beam, after being disturbed, should come to rest fairly quickly and without too many oscillations.

Satisfying completely any one of these conditions usually invalidates the other two and the successful balance represents a skilful compromise between all three. Some of these compromises are discussed in the next section.

Sensitivity of a Balance

The position of the beam is read by means of a pointer moving over a graduated scale, thus the sensitivity might be expressed as '1 mg per

scale division', meaning that one milligram will turn the beam so that the pointer moves on one division. More commonly it is given as x scale divisions per mg', thus a large figure indicates a high sensitivity. An expression for the sensitivity of a balance can be derived as follows.

Let the total masses in the pans be m and $(m + \delta m)$, and let this small difference turn the beam through an angle θ (Fig. 4.10). The beam has a mass M and its centre of gravity is at a distance h below the knife-edge; the arms of the beam are each of length d.

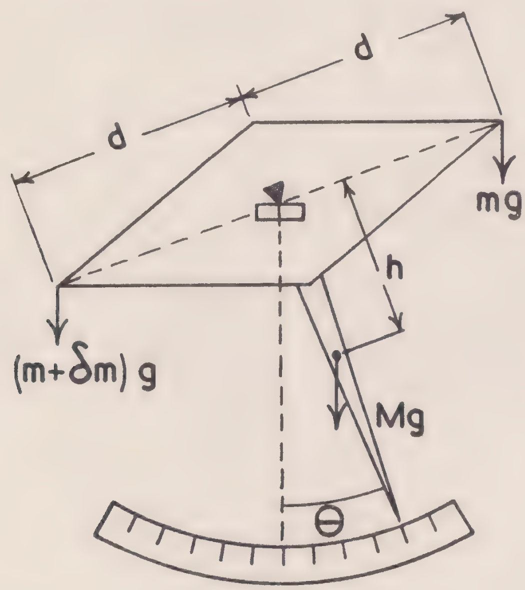


Fig. 4.10

Then, taking moments about the knife-edge, the total moment is given by:

$$\gamma = Mgh \sin \theta + mgd \cos \theta - (m + \delta m)gd \cos \theta \qquad (19)$$

If the beam is at rest, the total moment of the forces acting on it must be zero,

hence $Mgh \sin \theta = (m + \delta m)gd \cos \theta - mgd \cos \theta$ . . . (20)

or $Mh \sin \theta = \delta m \, d \cos \theta$,

$$\text{giving } \frac{\delta m}{\tan \theta} = \frac{Mh}{d} \qquad (21)$$

Now if θ is small, $\tan \theta \simeq \theta$, therefore (21) becomes:

$$\frac{\delta m}{\theta} = \frac{Mh}{d} \qquad (22)$$

Now $\delta m/\theta$ is one expression for the sensitivity of the balance (in 'grams per radian' if c.g.s. units are used), and knowing the geometry of any particular balance, this could be converted into 'milligrams per scale division'.

It is not often that one needs to *calculate* in this way the sensitivity of a balance; the figure is usually given by the maker, and if not, the deflection in scale divisions for a known small weight is readily observed. However, the expression gives a useful guide when designing a sensitive balance. If the balance is to be very sensitive, then a very small weight must be sufficient to deflect the beam, i.e. the factor $\delta m/\theta$ must be as small as possible. This can be achieved if Mh/d is as small as possible and thus M and h must be small and d large.

The sensitive balance therefore needs a light beam with its centre of gravity close up to the knife-edge, also it must have very long arms. The beam, however, must have enough strength to support the load, so that its mass cannot be reduced too far, neither can the arms be made too long without increasing the mass unduly. Further, if the centre of gravity is near the knife-edge, the beam will be subjected to very little restoring force when it is disturbed and hence will swing only very slowly and take a long time to come to rest. The design of the balance involves a compromise between all of these factors, although in some of the better balances the time of vibration is reduced by fitting large vanes to the end of the beam and using the air resistance on these vanes to damp the motion of the beam.

It will be noticed that the sensitivity depends on the distance of the centre of gravity from the knife-edge, thus the sensitivity can be changed if this distance is a variable. Some balances have a small weight which can be moved up and down the pointer, or else a 'gravity ball' running on a screwed rod above the beam; moving either of these changes the position of the centre of gravity of the beam and so changes the sensitivity. With suitable manipulation it is possible to adjust the sensitivity so that 1 scale division means 1 milligram (or any other suitable quantity); thus, when weighing, the last figures of the answer can be read from the scale.

The above treatment has assumed that the bearing edges of the knife-edges are all in the same plane, but it is now common practice to build a balance with the outer knife-edges lower than the central one. An expression for the sensitivity, similar to the one given above, can be produced by a parallel argument, but in this case it will be found that the sensitivity is also dependent on the load carried by the pans.

4.7 Couples

So far, the effect of a single force applied to a body has been considered, and it has been seen that both translation and rotation are produced. If, however, two equal and opposite forces, acting in different lines, are applied to the body (Fig. 4.11), both have a moment about any point in the body and will tend to produce rotation, but the total

force acting in any direction is zero and hence the body will suffer no translatory motion. Such a system of forces is described as a *couple.*

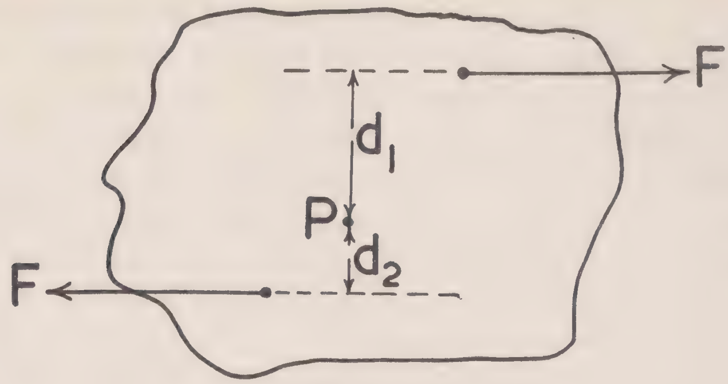


Fig. 4.11

A couple is the physical quantity that disturbs a body from its state of rotational rest just as a force disturbs a body from its state of linear rest; a couple is, in fact, the rotational counterpart of a force in linear motion. Many of the ideas and expressions for rotational motion can be remembered by analogy with the more familiar equations of linear motion as will be seen in the following pages.

The total moment of the couple shown in Fig. 4.11 about P is given by:

$$\begin{aligned}\gamma &= Fd_1 + Fd_2 \\ &= F(d_1 + d_2) \\ &= Fd \qquad (23)\end{aligned}$$

where d is the total perpendicular distance between the two forces.

The moment of a couple is more usually called a *Torque* and for this the symbol Γ (Greek capital 'gamma') is used. The torque of a couple is thus independent of the choice of position of the point P and is always given by the product of one of the forces and the perpendicular distance between them.

$$\text{Thus } \Gamma = F \times d \qquad (24)$$

The unit of torque will obviously be the unit of moment, i.e. the dyne.cm in the c.g.s. system while the lb-wt.ft is the gravitational unit in the f.p.s. system. Torque is a vector quantity, the direction of the vector being along the axis of the twist produced by the torque.

4.8 Equilibrium

When a body is at rest under the action of a set of forces, it is said to be in equilibrium; various classes of equilibrium can be distinguished depending upon whether the body, if subjected to a slight disturbance, tends either to return to its original position or to move farther away from that position.

The classical example is a cone which could, with care, be poised on its apex. The forces then acting upon it are the weight of the cone

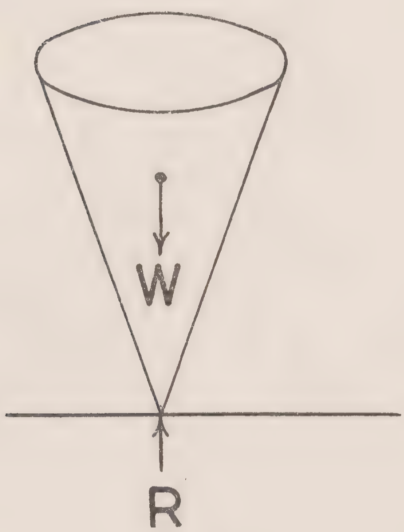


Fig. 4.12

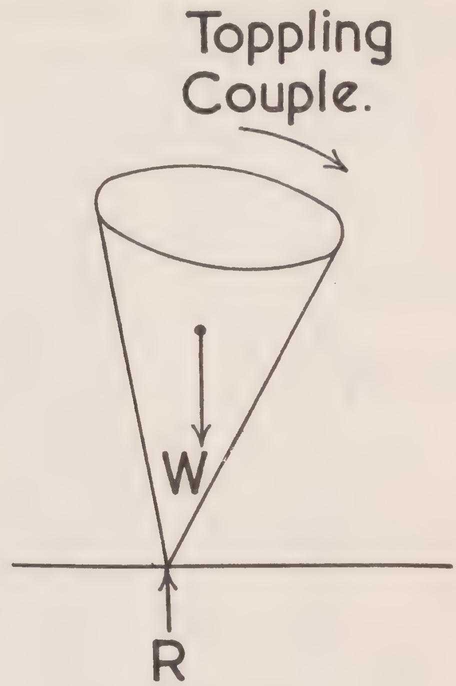


Fig. 4.13

vertically downwards and the reaction upwards of the plane on which it stands. When these two forces are in the same straight line, they exert no couple on the cone, which therefore remains balanced in this position

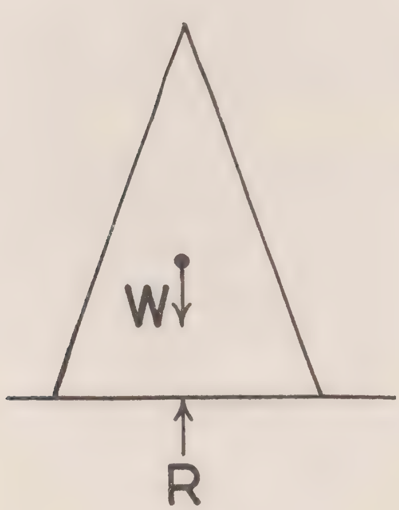


Fig. 4.14

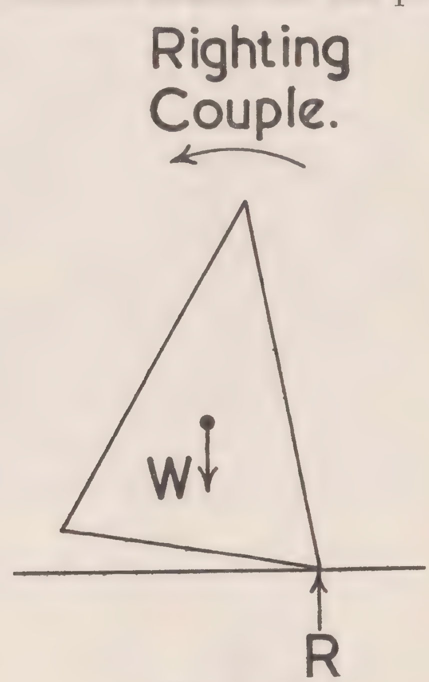


Fig. 4.15

(Fig. 4.12). If, however, the system is disturbed slightly, these forces form a couple which tends to move the cone farther away from its balance position (Fig. 4.13) and it falls over; although Fig. 4.12 repre-

sents a position of equilibrium, it is referred to as a position of *Unstable Equilibrium.*

Alternatively, if the cone stands on its base (Fig. 4.14), it is again in equilibrium, but this time a *small* disturbance (Fig. 4.15) transfers the point of contact, and hence the reaction, to one edge of the base of the cone. The weight and the reaction now form a couple acting so as to restore the cone to its former position. This is called a position of *Stable Equilibrium.*

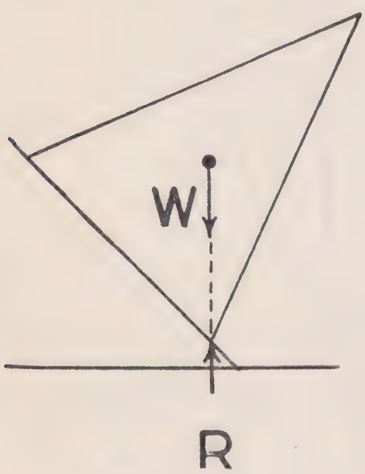


Fig. 4.16

It will be noticed that the cone can in fact be tilted until the centre of gravity lies vertically above the point of contact and still return to its original position when released; thus, if placed on a rough board, it could be tilted to the position shown in Fig. 4.16 before it would topple. This provides a further test for equilibrium in the case of a body having a base of finite area: stable equilibrium is possible if the *vertical* line drawn through the centre of gravity passes through the base. (Or through the area bounded by a piece of string drawn tightly around the base, if it is re-entrant in shape or the body has legs.)

Further, it will be noticed that if the equilibrium is stable, a small rotation has the effect of lifting the centre of gravity, while if the equilibrium is unstable, then the centre of gravity falls as the body rotates. This is again a general case—if the centre of gravity of a body is in its lowest possible position, then the equilibrium of the body is stable; the equilibrium is unstable if the centre of gravity is at its highest point. The foregoing also means that the potential energy of a system is at a minimum value when in a stable position and at a maximum in the unstable position.

There remains one further kind of equilibrium—that displayed by a sphere on a horizontal plane, when the sphere will stay in any position. This is called *Neutral Equilibrium* and occurs whenever the centre of gravity must, because of the geometry of the system, remain vertically above the point of contact, hence the body experiences no couple how-

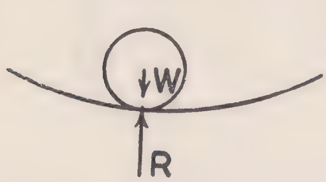


Fig. 4.17

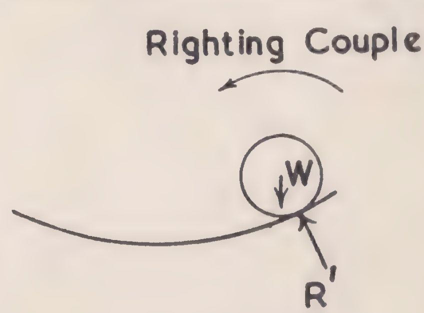


Fig. 4.18

ever it may be displaced. The position of the centre of gravity in this case is at a fixed height and the potential energy of the system is unchanged by a displacement.

It must be emphasised that when considering the equilibrium of a body, the shape of the surface on which it stands must also be considered; for example, a sphere is in neutral equilibrium on a horizontal plane surface but in stable equilibrium at the lowest point of a concave surface as shown in Fig. 4.17 and Fig. 4.18.

The paragraphs above refer to the stability of bodies when at rest or in static equilibrium; another form of stability, called dynamic equilibrium, applies to bodies in motion; a top is a familiar example of this, for, whilst spinning, it is capable of remaining indefinitely in an upright position which would otherwise be unstable. Considerations of dynamic stability are in general beyond the scope of this book.

SUMMARY OF TESTS WHICH CAN BE USED TO ASSESS THE NATURE OF AN EQUILIBRIUM POSITION

In all tests, imagine the body to be displaced slightly from the equilibrium position to be examined.

Test	*Stable*	*Unstable*	*Neutral*
Couple	Restoring couple produced	Toppling couple produced	No couple
Vertical through centre of gravity	Still passes through base	Falls outside base	Still goes through point of contact
Position of centre of gravity	Raised from lowest position	Lowered from highest position	No change in height
Potential energy	Increased from minimum value	Decreased from maximum value	Unchanged

4.9 Work done by a Couple

It has already been seen that a couple is the rotational counterpart of a force, while an angular displacement, i.e. a twist, is obviously the rotational counterpart of a linear displacement. Now if a force causes a linear displacement, it does work; similarly, then, if a couple causes a twist we shall also expect it to do work. The work can be calculated as follows.

Let the couple consist of two parallel forces F applied at A and B and let the body be free to rotate about the point P (Fig. 4.19). Let the body rotate through small angle $\delta\theta$ under the action of the couple, then

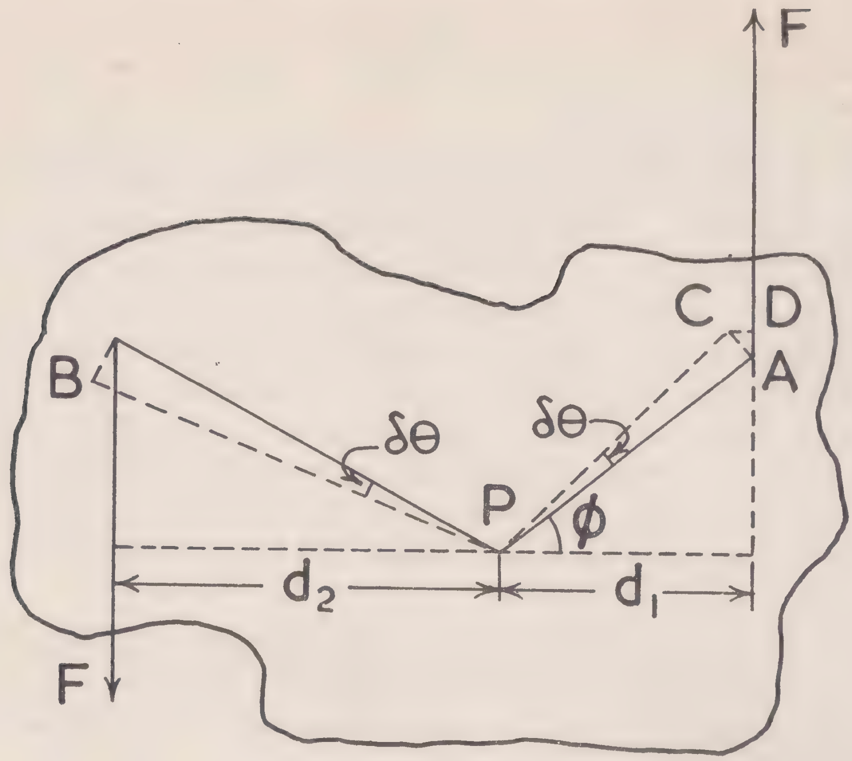


Fig. 4.19

the point of application of the right-hand force moves through a distance $PA \,.\, \delta\theta$ perpendicular to the line PA.

$$\text{Thus } AC = PA \,.\, \delta\theta \qquad (25)$$

Now the distance moved parallel to the force F is given by:

$$\begin{aligned} AD &= AC \sin \phi \\ &= PA \sin \phi \, \delta\theta, \\ \text{but } d_1 &= PA \sin \phi, \\ \text{thus } AD &= d_1 \,.\, \delta\theta \end{aligned} \qquad (26)$$

This is the distance moved by the point of application of the force parallel to itself, thus the work done by this force during the small twist is given by:

$$\delta W = F \,.\, d_1 \,.\, \delta\theta \qquad (27)$$

By a similar argument, the other force does work equal to $Fd_2\delta\theta$ and the total work done by both forces is given by:

$$\begin{aligned} \delta W &= F(d_1 + d_2)\delta\theta \\ &= Fd \,.\, \delta\theta, \end{aligned}$$

where d is the perpendicular distance between the two forces.
But Fd is the torque Γ of the couple, thus

$$\delta W = \Gamma \delta\theta \qquad (28)$$

If this couple causes a finite twist θ, the total work can be found by

dividing the twist into a number of indefinitely small elements and then integrating. The total work is then given by

$$W = \int_0^\theta \Gamma d\theta$$

$$= \Gamma\theta \quad . \quad . \quad . \quad . \quad . \quad (29)$$

if the couple has a constant value.

Thus work done by a constant couple = Torque of couple × Twist produced by couple.

Note that for theoretical purposes angular displacement must be measured in circular measure, i.e. radians, and not in degrees.

Example 1. *The crankshaft of a car engine rotates at* 4000 *rpm and transmits* 15 *hp to the roadwheels. Find the torque in the shaft.*

Work done in 1 sec = 15 × 550 ft.lb-wt.

But shaft makes $\frac{4000}{60}$ rev. per sec and therefore turns through $\frac{4000 \times 2\pi}{60}$ radians in one second, which reduces to $\frac{400\pi}{3}$ radians in one second.

If torque in shaft is Γ lb-wt.ft, work done by shaft in one second is $\frac{400\pi\Gamma}{3}$ ft.lb-wt,

$$\text{thus } \frac{400\pi\Gamma}{3} = 15 \times 550$$

$$\text{or } \Gamma = \frac{3 \times 15 \times 550}{400\pi} \text{ lb-wt.ft}$$

$$= 19{\cdot}7 \text{ lb-wt.ft.}$$

4.10 Angular Velocity

Continuing the analogy between linear and rotational motion, angular velocity can next be defined in a similar fashion to linear velocity. If a body is rotating about an axis through the point P perpendicular

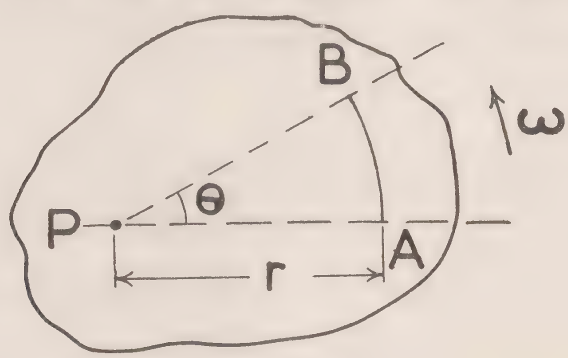


Fig. 4.20

to the plane of rotation (Fig. 4.20) so that a line drawn on it through P turns through equal angles in equal increments of time, then it is said

to rotate with a constant *angular velocity*. The symbol normally used for angular velocity is ω, thus if the radius PA (Fig. 4.20) turns through an angle θ to the position PB in time t,

$$\omega = \frac{\theta}{t}.$$

If the angular velocity is not constant, then its value at any instant is given by

$$\omega = \frac{d\theta}{dt} \qquad . \quad . \quad . \quad . \quad . \quad (30)$$

The magnitude of the angle θ (Fig. 4.20) is given in radians by (arc $AB)/r$; this is the ratio of two similar quantities, hence angle is a pure number and has no dimensions.

The dimensions of angular velocity are given by

$$[\omega] = \left[\frac{\text{angle}}{\text{time}}\right] = [T^{-1}].$$

The unit in which angular velocity is measured is the radian.sec^{-1} in the c.g.s., f.p.s. or M.K.S. systems of units.

Angular velocity is a vector quantity, the direction of the vector once again being along the axis of rotation and given by the right-hand corkscrew rule.

The linear velocity of any point on a rotating body can be related to its angular velocity as follows.

If the body of Fig. 4.20 is imagined to make an indefinitely small rotation $d\theta$ and the arc AB to be of length ds, then $d\theta = ds/r$; substituting this in Equation (30) gives:

$$\omega = \frac{1}{r}.\frac{ds}{dt}.$$

But ds/dt is the linear velocity of the point A, and is perpendicular in direction to the radius PA, or tangential to the path taken by A, thus the tangential velocity v_t of A is given by:

$$v_t = \omega r \qquad . \quad . \quad . \quad . \quad . \quad (31)$$

Example 2. *A stone is whirled in a vertical plane at the end of a string $2\frac{1}{2}$ ft long. The string breaks when it is horizontal and the stone rises vertically a maximum distance of* 40 *ft. Find the angular velocity of the string immediately before it broke.*

The stone rises against the action of gravity, thus if its initial velocity is u ft.sec^{-1}, u is given by:

$$0 = u^2 - 2 \times 32{\cdot}2 \times 40,$$

$$\text{or } u = 50{\cdot}6 \text{ ft.sec}^{-1}.$$

Angular velocity of the string is given by u/r where r is the length of the string,

$$\text{thus } \omega = \frac{50 \cdot 6}{2 \cdot 5} \text{ radian.sec}^{-1}$$

$$= \frac{50 \cdot 6}{2 \cdot 5 \times 2\pi} \text{ rev. per second}$$

$$= 3 \cdot 2 \text{ rev. per second.}$$

4.11 Angular Acceleration

If the angular velocity of a body varies, then it suffers an angular acceleration. Following again the analogy of linear motion, angular acceleration is defined as the rate of change of angular velocity; the symbol α is normally used, thus:

$$\alpha = \frac{d\omega}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

which can also be written as:

$$\alpha = \frac{d^2\theta}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

$$\text{since } \omega = \frac{d\theta}{dt}.$$

The dimensions of angular acceleration are given by:

$$[\alpha] = \left[\frac{\text{Angular Velocity}}{\text{Time}}\right]$$

$$= \left[\frac{T^{-1}}{T}\right]$$

$$= [T^{-2}].$$

The unit in which angular acceleration is measured is the radian.sec^{-2} in each of the three common systems of units.

Angular acceleration is also a vector quantity, the direction of the vector again being along the axis of rotation.

The tangential velocity of any point A on a rotating body has already been found, Equation (31), to be given by $v_t = \omega r$, and is, of course, constant so long as ω has a steady value. If, however, ω varies, then the point has a tangential acceleration given by the rate of change of tangential velocity, i.e.

$$\text{Tangential acceleration} = \frac{dv_t}{dt}$$

$$= r \cdot \frac{d\omega}{dt} \quad . \quad . \quad (34)$$

since r is a constant.

This equation can also be written as:

$$\text{Tangential acceleration} = r\alpha$$

$$= \frac{rd^2\theta}{dt^2}.$$

4.12 Rotational Motion under a Constant Angular Acceleration

If a body is subjected to a constant angular acceleration, then a set of equations can be built up similar to those given on page 9 for linear motion. The method of derivation is exactly the same and so is not given here.

If the body, subjected to an angular acceleration α, increases its angular velocity from ω_1 to ω_2 in time t and also turns through an angle θ, then

$$\omega_2 = \omega_1 + \alpha t \text{ (cf. Chapter 1, Equation (7))} \qquad (35)$$

$$\theta = \omega_1 t + \tfrac{1}{2}\alpha t^2 \text{ (Chapter 1, Equation (8))} \qquad (36)$$

$$\omega_2^2 = \omega_1^2 + 2\alpha\theta \text{ (Chapter 1, Equation (9))} \qquad (37)$$

$$\theta = \omega_2 t - \tfrac{1}{2}\alpha t^2 \text{ (Chapter 1, Equation (10))} \qquad (38)$$

$$\theta = \frac{\omega_1 + \omega_2}{2} . t \text{ (Chapter 1, Equation (11))} \qquad (39)$$

4.13 Moment of Inertia

The analogy between couple and force has already been noticed; it is natural to expect, then, that the application of a couple to a body will produce in it an angular acceleration just as a force produces linear acceleration. This angular acceleration can be calculated as follows.

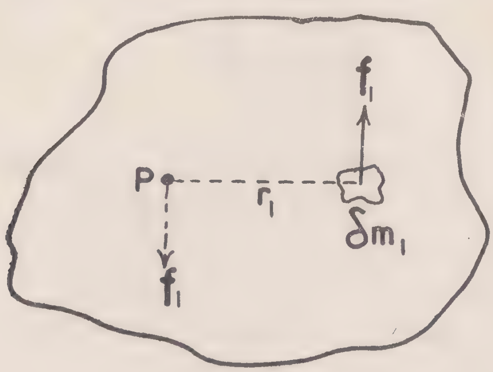


Fig. 4.21

Let a body be subjected to a total couple Γ; this couple will normally be applied to the body as a whole, but we can imagine the couple divided into small portions, each part being applied to an element of the body. For example, in Fig. 4.21 the element of mass δm_1 has applied to it a force f_1 which, together with the reaction at the pivot, contributes an element of couple $\delta\Gamma$ given by

$$\delta\Gamma = f_1 r_1.$$

Now the mass δm_1 has applied to it a force f_1 and hence moves with a linear acceleration a_1 (perpendicular to the radius r_1) given by:

$$f_1 = \delta m_1 a_1.$$

The linear acceleration is related to the angular acceleration α by the equation:

$$a_1 = r_1\alpha$$

$$\text{thus } f_1 = \delta m_1 r_1 \alpha \qquad (40)$$

[Note that α must be the same for all points on the body since it rotates as a whole,

$$\text{i.e. } \frac{f_1}{\delta m_1 r_1} = \alpha = \text{constant at any instant.}$$

The value of f to be applied to any element must thus be chosen so as to satisfy this equation.]

$$\text{Now } \delta\Gamma = f_1 r_1$$

$$= \delta m_1 r_1^2 \alpha \text{ (from Equation 40)}$$

and thus the total couple applied to the body is given by:

$$\Gamma = \Sigma(r^2\delta m)\alpha \qquad (41)$$

where the summation has to be taken over all elements of the body.

The quantity $\Sigma(r^2\delta m)$ is called the *Moment of Inertia* of the body, symbol I, thus:

$$\Gamma = I\alpha \qquad (42)$$

$$\text{or } \alpha = \Gamma/I$$

Equation (42) can be compared with the equation for linear motion:

$$f = ma,$$

from which it is seen that moment of inertia is the rotational counterpart of mass, i.e. the property of a body which makes it resist the attempt of a couple to establish rotational motion.

The dimensions of moment of inertia can be calculated from the equation:

$$I = \Sigma(r^2\delta m).$$

$$\text{Thus } [I] = [r^2] \times [\delta m]$$

$$= [L^2] \times [M]$$

$$= [ML^2].$$

The c.g.s. unit of moment of inertia is the gm.cm^2 and the f.p.s. unit is the lb.ft^2.

Moment of inertia is not a vector, consequently, if a body has a number of parts, the moment of inertia of each part about the axis of rotation can be found separately and then the results added arithmetically.

The moment of inertia of a body is found by evaluating $\Sigma(r^2\delta m)$ for all the elements in the body. This is usually done by dividing the body into a number of small elements, summing, and obtaining the limit

which this sum approaches as the number of elements increases indefinitely, the size of each element correspondingly becomes vanishingly small. This is precisely the process known as integration and hence the sum may be replaced by an integral, thus:

$$I = \int r^2 dm,$$

where the limits of the integral are chosen to include the whole body. For details of this process, the student is referred to a book on Applied Mathematics.

4.14 Radius of Gyration

The dimensions of moment of inertia are $[ML^2]$, thus the moment of inertia of any body can always be expressed as the product of the mass of the body and the square of a length, i.e.

$$I = mk^2,$$

where k is a length called the *Radius of Gyration* of the body.

This length depends on the distribution of mass in the body; for example, the moment of inertia of a rod about an axis perpendicular to its length through the centre of gravity is given by:

$$I = \frac{ml^2}{3}.$$

Comparing this with $I = mk^2$ indicates that:

$$k = \frac{l}{\sqrt{3}},$$

or the radius of gyration of a rod of length $2l$ about an axis through its centre of gravity is $\frac{l}{\sqrt{3}}$.

The radius of gyration of a disc of radius r is equal to $\frac{r}{\sqrt{2}}$; there is no general rule for finding radii of gyration, each case has to be worked out individually. The radius of gyration of a heavy hoop about an axis through the centre of the hoop and perpendicular to its plane is equal to the radius of the hoop itself. This provides some idea of the physical meaning of radius of gyration, for if the material of any body is formed into a hoop having the same moment of inertia as the original body, then the radius of the hoop is the radius of gyration of the body.

4.15 Parallel and Perpendicular Axes Theorems for Moments of Inertia

Parallel axes theorem.

If a body of mass m rotates about an axis passing through a point P,

it can be shown that its moment of inertia about this axis is given by

$$I_p = I_c + ml^2 \qquad . \quad . \quad . \quad . \quad (43)$$

where I_c is the moment of inertia of the body about a parallel axis through the centroid and l is the perpendicular distance between the two axes.

It should be noted that $I_c = mk^2$, where k is the radius of gyration about the centre of gravity; thus Equation (43) can be written as:

$$I_p = mk^2 + ml^2 \qquad . \quad . \quad . \quad . \quad (44)$$

$$= m(k^2 + l^2) \qquad . \quad . \quad . \quad . \quad (45)$$

Perpendicular axes theorem.

If the moments of inertia of a lamina about two perpendicular axes in the plane of the lamina are known, then the sum of these gives the moment of inertia of the lamina about the axis perpendicular to its plane and passing through the point of intersection of the two original axes,

$$\text{or } I_x + I_y = I_z \qquad . \quad . \quad . \quad . \quad (46)$$

It should be noted that this theorem *applies only to a lamina*, whereas the parallel axes theorem can be applied to any shape of body.

With these two theorems and the knowledge of a few standard moments of inertia, the moment of inertia of almost any body of regular shape can be found.

4.16 Kinetic Energy of a Rotating Body

If the body of Fig. 4.22 is rotating about an axis through P, then every element of the body will have a linear velocity at any instant and as a result will possess an amount of kinetic energy equal to $(\frac{1}{2}\delta mv^2)$, where δm and v are the mass and linear velocity of the element respectively.

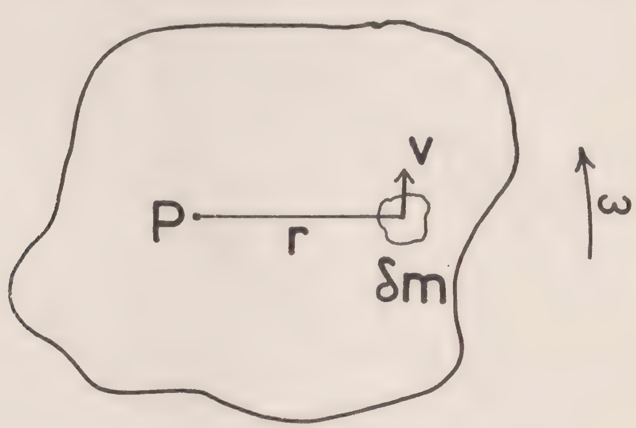


Fig. 4.22

Now $v = \omega r$ if ω is the angular velocity of the body and r the distance of the element from P; thus the kinetic energy of the element is given by $(\frac{1}{2}\delta m \, . \, \omega^2 r^2)$.

Kinetic energy is a scalar quantity, hence the total energy of the rotating body can be found merely by summing the separate energies of the elements, thus:

$$\text{Total kinetic energy} = \tfrac{1}{2}\omega^2\Sigma(\delta mr^2),$$

but $\Sigma(\delta mr^2)$ is the moment of inertia of the body about P, hence:

$$\text{Kinetic energy of rotating body} = \tfrac{1}{2}I\omega^2 \quad . \quad . \quad (47)$$

(The correspondence with linear motion where kinetic energy $= \frac{1}{2}mv^2$ will be noted.)

The dimensions of this expression agree with those previously obtained for energy, thus:

$$[\tfrac{1}{2}I\omega^2] = [ML^2 . (T^{-1})^2]$$
$$= [ML^2T^{-2}],$$

which are the dimensions of energy (see page 28).

If moment of inertia is measured in gm.cm^2 and angular velocity in radian.sec^{-1}, then the energy appears in ergs. When the f.p.s. system of units is used, then moment of inertia is given in lb.ft^2, the unit of angular velocity is still the radian.sec^{-1}, and the energy is given in ft.poundals; the engineer, however, normally measures energy in the gravitational unit, ft.lb-wt (equal to 32·2 ft.poundals), and therefore must measure moment of inertia in lb-wt.ft^2 (equal to 32·2 lb.ft^2) although moment of inertia is a function of the mass of a body and is in no way connected with its weight.

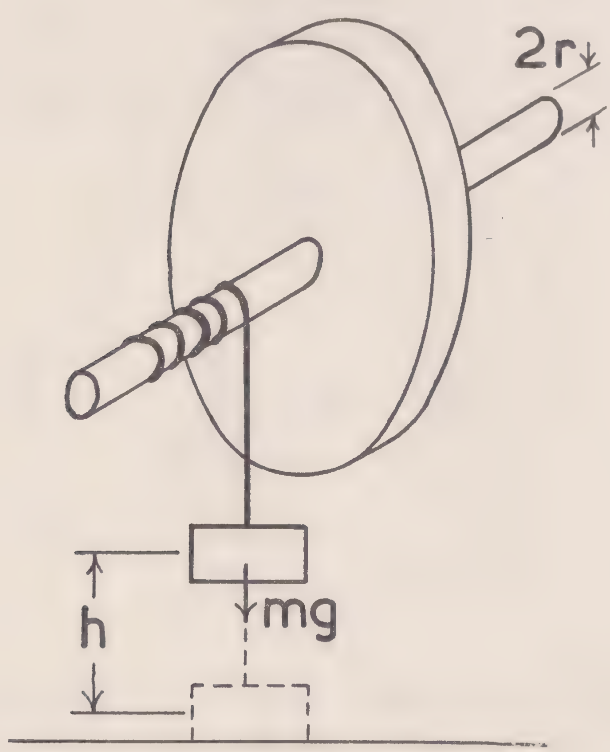


Fig. 4.23

4.17 Experimental Method of Measuring Moment of Inertia

The moment of inertia of a symmetrical body, such as a flywheel mounted on an axle, can be measured by hanging a weight on a cord wrapped around the axle as shown in Fig. 4.23: as the weight falls, the wheel acquires an angular acceleration and by measuring this acceleration the moment of inertia of the wheel can be calculated as follows.

If the weight and wheel both start from rest and by the time the weight reaches the floor it has acquired a velocity v while the wheel has an angular velocity ω, then:

$$\text{Kinetic Energy gained by wheel} = \tfrac{1}{2}I\omega^2,$$

$$\text{and kinetic energy gained by weight} = \tfrac{1}{2}mv^2,$$

$$\text{thus total gain in kinetic energy} = \tfrac{1}{2}(I\omega^2 + mv^2).$$

But the potential energy lost by the weight is mgh if it falls through a distance h, and since the energy of the system must remain constant, this loss must equal the gain in kinetic energy.

$$\text{Hence } \tfrac{1}{2}(I\omega^2 + mv^2) = mgh.$$

$$\text{Thus } I = \frac{2mgh - mv^2}{\omega^2}.$$

Now $v = \omega r$ where r is the radius of the axle thus:

$$I = mr^2\left[\frac{2gh}{v^2} - 1\right],$$

also, if t is the time taken by the weight to fall to the floor, then:

$$h = \frac{v}{2} \cdot t,$$

$$\text{thus } I = mr^2\left[\frac{gt^2}{2h} - 1\right] \qquad (48)$$

from which the moment of inertia of the wheel can be calculated.

No account of the frictional effects at the bearings has been taken here; various experimental methods are available for allowing for friction and for these the reader is referred to a textbook on practical physics.

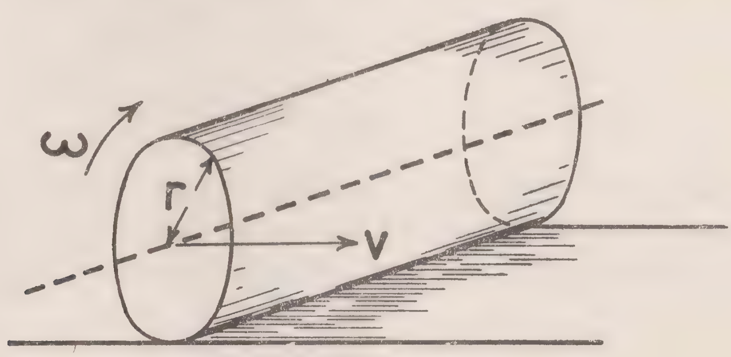


Fig. 4.24

4.18 Kinetic Energy of a Rolling Body

If a body, such as a cylinder (Fig. 4.24), is rolling along, then it has kinetic energy both of rotation and translation. In this case we have:

$$\text{Rotational kinetic energy} = \tfrac{1}{2}I\omega^2.$$

$$\text{Translational kinetic energy} = \tfrac{1}{2}mv^2.$$

$$\text{Hence total kinetic energy} = \tfrac{1}{2}(I\omega^2 + mv^2).$$

But $v = \omega r$ since the body is instantaneously rotating about the point of contact with the plane.

$$\text{Thus total kinetic energy} = \tfrac{1}{2}\left\{\frac{Iv^2}{r^2} + mv^2\right\}.$$

Also $I = mk^2$ where k is the radius of gyration about the axis.

$$\text{Thus total kinetic energy} = \tfrac{1}{2}v^2\left\{\frac{mk^2}{r^2} + m\right\}$$

$$= \tfrac{1}{2}mv^2\left\{1 + \frac{k^2}{r^2}\right\} \quad . \quad . \quad (49)$$

Body Rolling Down an Incline

If a body is released from rest at A on an inclined plane and rolls down to B, thereby acquiring a linear velocity v (Fig. 4.25), then the gain in energy, from Equation (49), is given by:

$$\text{Kinetic energy gained} = \tfrac{1}{2}mv^2\left\{1 + \frac{k^2}{r^2}\right\}.$$

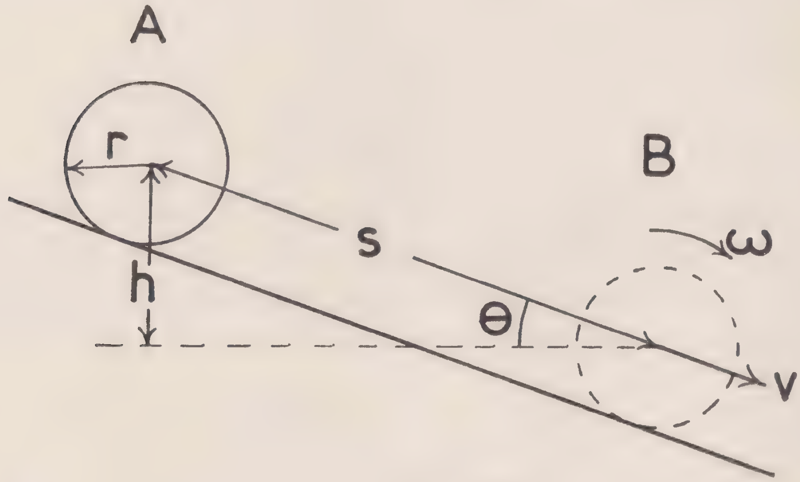


Fig. 4.25

But if the body falls through a vertical height h, then the loss of potential energy is mgh. Since the energy of the system must remain constant, this loss must be equal to the gain in kinetic energy.

$$\text{Hence } \tfrac{1}{2}mv^2\left\{1 + \frac{k^2}{r^2}\right\} = mgh,$$

$$\text{and } mk^2 \cdot \frac{v^2}{r^2} = 2mgh - mv^2$$

$$\text{or } I = \frac{2mghr^2}{v^2} - mr^2,$$

$$\text{since } mk^2 = I$$

$$\text{thus } I = mr^2\left[\frac{2gh}{v^2} - 1\right].$$

But if t is the time taken to roll from A to B, then $s = \dfrac{vt}{2}$

$$\text{and } I = mr^2\left[\frac{ght^2}{2s^2} - 1\right].$$

Also $h = s \sin\theta$ if θ is the slope of the plane.

$$\text{Thus } I = mr^2\left[\frac{gt^2 \sin\theta}{2s} - 1\right] \quad . \quad . \quad (50)$$

from which the moment of inertia of the body can again be calculated.

If the body is not of a shape conducive to rolling, its moment of inertia may still be found by attaching to it a cylindrical axle passing through the centre of gravity of the body. The composite body is then timed whilst rolling on the axle down a pair of inclined rails; from this the moment of inertia of the whole body can be found and the moment of inertia of the axle subtracted from it. Further details of this experiment will be found in *Experimental Physics*, by Daish and Fender, published by the English Universities Press, Ltd.

4.19 Angular Momentum

Couple and angular acceleration are related by the expression:

$$\Gamma = I\alpha \text{ (see Equation (42)).}$$

This can be written as:

$$\Gamma = I\frac{d\omega}{dt}$$

$$= \frac{d}{dt}(I\omega) \quad . \quad . \quad . \quad . \quad . \quad (51)$$

from which it is seen that torque is equal to the rate of change of the quantity $(I\omega)$; now torque is the rotational counterpart of force, and force is equal to the rate of change of momentum; the quantity $(I\omega)$ is obviously, then, the rotational counterpart of momentum and is called *Angular Momentum*; no especial symbol is used for it.

The dimensions of angular momentum are given by:

$$[I\omega] = [ML^2 \times T^{-1}],$$
$$= [ML^2T^{-1}],$$

and thus the c.g.s. unit in which it is measured is the gm.cm^2.sec^{-1}, the corresponding f.p.s. unit is the lb.ft^2.sec^{-1} and the gravitational unit lb-wt.ft^2.sec^{-1} is also used.

Equation (51) can be written as:

$$\Gamma dt = I d\omega$$

if the moment of inertia I is a constant. Thus, if the angular velocity changes from ω_1 to ω_2 in time t, we have

$$\int_0^t \Gamma dt = \int_{\omega_1}^{\omega_2} I d\omega,$$
$$= I\omega_2 - I\omega_1 \quad . \quad . \quad . \quad . \quad . \quad (52)$$

Now $I\omega_2 - I\omega_1$ is the change in angular momentum produced by the couple Γ in time t. On comparing this with the linear case, $\int_0^t \Gamma dt$ is recognised as the rotational counterpart of an impulse.

It is called an *Angular Impulse* or *Impulsive Moment*, and thus:

Angular Impulse = Change in Angular Momentum.

If the torque Γ is constant throughout the angular impulse, then:

$$\int_0^t \Gamma dt = \Gamma t.$$

4.20 Uniform Motion in a Circle

The case of a body performing a steady motion in a circle represents a particular type of rotational motion very frequently met in practice. Before the physics of this type of motion is considered, some theoretical ideas are first needed.

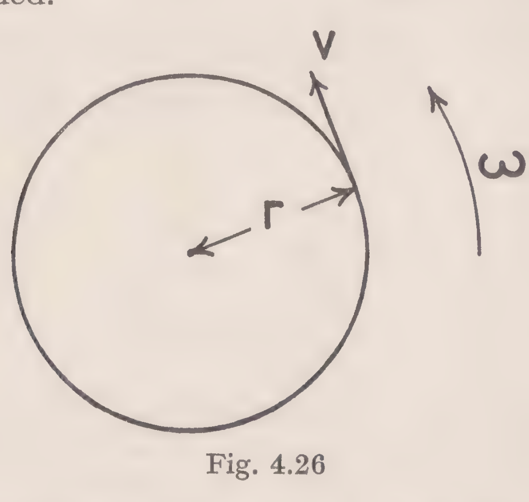


Fig. 4.26

If a particle is moving steadily around a circle of radius r with a constant angular velocity ω (Fig. 4.26), then its motion can be resolved at

any point into two components directed along the radius and tangent respectively, and the velocity and acceleration in these directions calculated.

Tangential Velocity

It has already been seen (page 72) that the tangential velocity v is given by

$$v = r\omega.$$

Radial Velocity

The radial velocity is given by the rate of change of the length of the radius, but the radius is of constant length if the path is a circle, hence its rate of change is zero and consequently the radial velocity is zero as well.

Tangential Acceleration

The tangential acceleration is given by the rate of change of tangential velocity, $r\omega$,

$$\text{i.e. Tangential Acceleration} = \frac{d}{dt}(r\omega),$$

but r and ω are both constants if the particle is moving in a circle with constant angular velocity.

Thus $\frac{d}{dt}(r\omega) = 0$ and there is no tangential acceleration.

Radial Acceleration

At first one might wonder if a radial acceleration does exist in circular motion since the radial velocity is constant (equal to zero). The tangential velocity is also constant in magnitude, but it changes its direction continually and it will be seen below that this change is due to a radial acceleration.

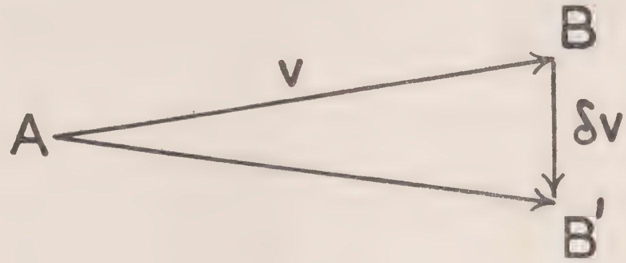


Fig. 4.27

If the velocity v of a particle performing any sort of motion can be represented at some instant by the vector AB (Fig. 4.27) and at some time δt later is of the same magnitude but in another direction represented by AB', then the change in velocity in the time δt is given by the

vector BB'. If this change in velocity is called δv, then the particle has been subjected to an average acceleration $\frac{\delta v}{\delta t}$ acting in the direction BB'.

If the time interval δt is now made indefinitely short, the angle θ between AB and AB' will also become indefinitely small and BB' will tend to be perpendicular to AB; thus the direction (but not the magnitude) of the velocity of a particle can be changed by a small amount by applying to it for a very short period an acceleration at right angles to the direction of the velocity.

If, now, this velocity is considered to be the tangential velocity of a particle moving in a circle, then the continually changing direction of this velocity can be accounted for by a radial acceleration. The magnitude of the acceleration can be calculated as follows.

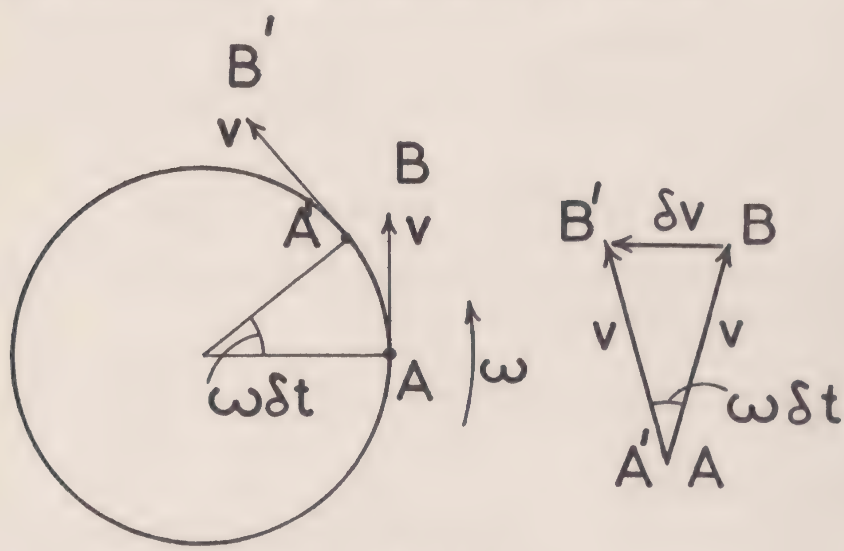


Fig. 4.28

If the particle moves from A to A' (Fig. 4.28) in time δt, then its velocity at the beginning and end of this period is given by the vectors AB and $A'B'$ respectively. Transferring these vectors to the right-hand diagram shows that the change in velocity δv is given by the vector BB'.

Now the angle between the two vectors AB and $A'B'$ is $\omega\delta t$ if ω is the angular velocity with which the particle performs its circular motion, therefore, considering BB' as a small arc of a circle centre A, we have:

$$\begin{aligned} \delta v &= BB' \\ &= AB \,.\, \omega\delta t \\ &= v\omega\delta t \end{aligned}$$

$$\text{or } \frac{\delta v}{\delta t} = v\omega.$$

Now $\delta v/\delta t$ is the average acceleration over the period δt, and the radial

acceleration is the value of this acceleration when δt is indefinitely small,

$$\text{i.e. radial acceleration} = \lim_{\delta t \to 0} \frac{\delta v}{\delta t}$$

$$= \lim_{\delta t \to 0} v\omega,$$

$$\left.\begin{aligned} \text{thus: radial acceleration} &= v\omega \\ &\left.\begin{aligned} &= \omega^2 r \\ &= \frac{v^2}{r} \end{aligned}\right\} \text{since } v = \omega r \end{aligned}\right\} \quad . \quad (53)$$

4.21 Centripetal and Centrifugal Force

A particle of mass m, moving steadily in a circular path, has a radial acceleration $\omega^2 r$ directed towards the centre of the circle; in order to give the particle this acceleration, a force must be impressed upon it, i.e. a particle will move in a circle only if constrained to do so by a force of the correct magnitude directed towards the centre of the circle. This force is called the *Centripetal* force and its magnitude is given by the product of the mass of the particle and its radial acceleration, thus:

$$\left.\begin{aligned} \text{Centripetal force} &= m \,.\, \omega^2 r \\ &= mv\omega \\ &= \frac{mv^2}{r} \end{aligned}\right\} \quad . \quad . \quad . \quad (54)$$

The force necessary to keep the body moving in a circle must be provided by some sort of constraint, i.e. the tension in the string in the case of a stone whirled on the end of a string, or the force exerted by the rails on the flanges of the wheels of an engine as it rounds a bend; but as the constraint exerts a centripetal force on the body, so the body must exert an equal and opposite force on the constraint, i.e. the stone pulls outwards on the string or the engine forces the rails outwards; this outward force is called the *Centrifugal* force.

We therefore have the centripetal force acting inwards on a body describing a circle, while the centrifugal force acts outwards on the constraint. It is possible to solve most problems in a slovenly fashion using the centripetal or centrifugal force indiscriminately, but it is better in every case to distinguish clearly between them.

Example 3.

(i) The Conical Pendulum

A bob of mass m hangs on a string of length l and describes a horizontal circle of radius r with a velocity v, so that the string sweeps out the surface of a

cone of semi-angle θ as shown in the diagram. Find the period of the pendulum.

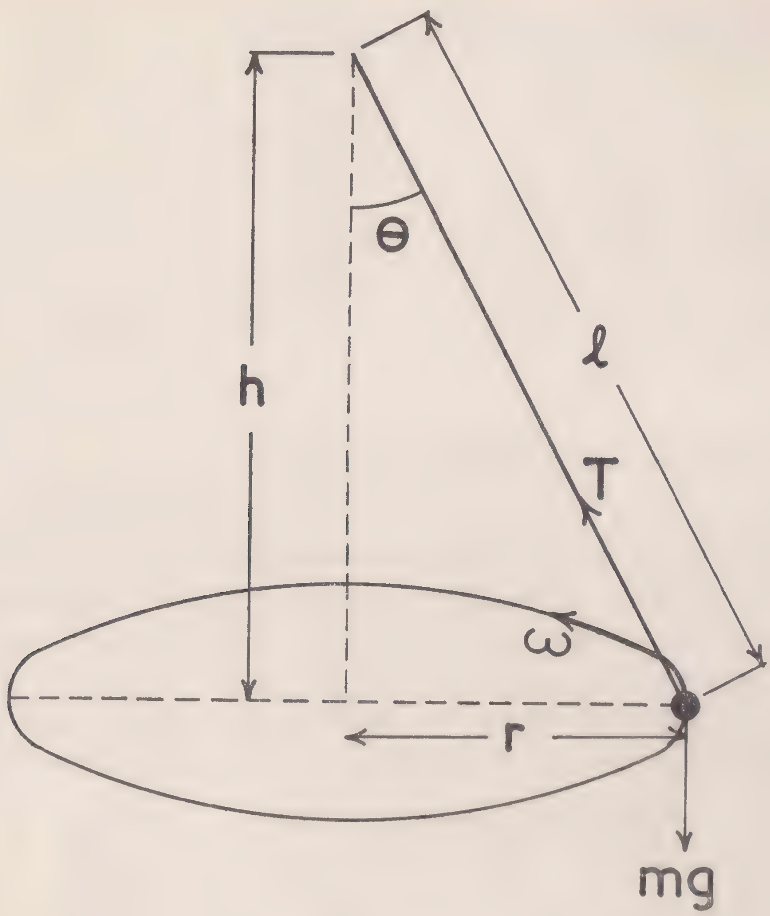


Fig. 4.29

The actual forces applied to the bob are the tension T in the string and the weight of the bob mg.

The force needed to make the bob run around a circle is a centripetal force $\frac{mv^2}{r}$ directed towards the centre of the circle and this must be supplied by T and mg when compounded.

Now mg, being vertical, cannot contribute towards a horizontal force: therefore $\frac{mv^2}{r}$ must be supplied entirely by the horizontal component of T.

$$\text{i.e. } T \sin\theta = \frac{mv^2}{r}.$$

Also the vertical component of T must cancel out mg since the bob has no vertical motion.

$$\text{Thus } T \cos\theta = mg.$$

Eliminating T by dividing corresponding sides of these equations gives:

$$\tan\theta = \frac{v^2}{gr}$$

$$\text{or } v = (gr \tan\theta)^{\frac{1}{2}}.$$

Now the time taken by the bob to travel once around the circle, which may be called the period of the pendulum, is given by:

$$\begin{aligned}\tau &= \frac{2\pi r}{v}\\ &= \frac{2\pi r}{(gr\tan\theta)^{\frac{1}{2}}}\\ &= 2\pi\sqrt{\frac{r}{g\tan\theta}}\\ &= 2\pi\sqrt{\frac{h}{g}},\end{aligned}$$

where h is the height of the cone swept out by the pendulum,

$$\text{or } \tau = 2\pi\sqrt{\frac{l\cos\theta}{g}},$$

where l is the length of the string.

(ii) Vehicles Negotiating Corners

(a) Car on a Flat Road

If the vehicle is to be able to get round a curve at speed, then a centripetal force must be applied to it. On a flat road this force can be provided only by the friction between the wheels and road (it is shown as the force F applied at one wheel in Fig. 4.30, although in practice this total force would be shared between the four wheels), and the speed at which the bend can be negotiated safely depends on the magnitude of this force.

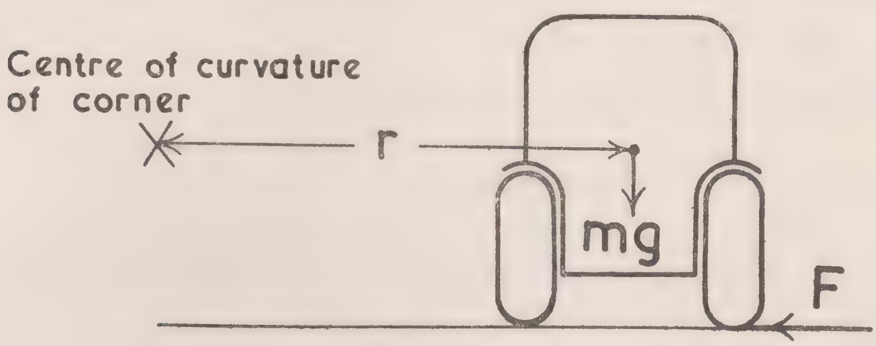


Fig. 4.30

The maximum value of F is $\mu_s.mg$ where μ_s is the limiting value of static friction between the wheels and the road, so that if $\mu_s\, mg < mv^2/r$, the frictional force will not be as large as the centripetal force needed to push the car around a bend of radius r at a speed v, and a skid towards the outside of the bend will occur.

The maximum speed for the bend is given by:

$$\frac{mv^2_{\max}}{r} = \mu_s\, mg$$

$$\text{or } v_{\max} = \sqrt{\mu_s\, rg} \qquad \qquad (55)$$

If $\dfrac{mv^2}{r} < \mu_s\, mg$, the car will avoid a skid, but yet another hazard faces the motorist—will the car overturn?

Suppose that the car is taking the bend so fast that the inner wheels are just lifting clear of the ground as in Fig. 4.31, then the forces acting on the car will be the weight mg vertically downwards through the centroid, a reaction equal

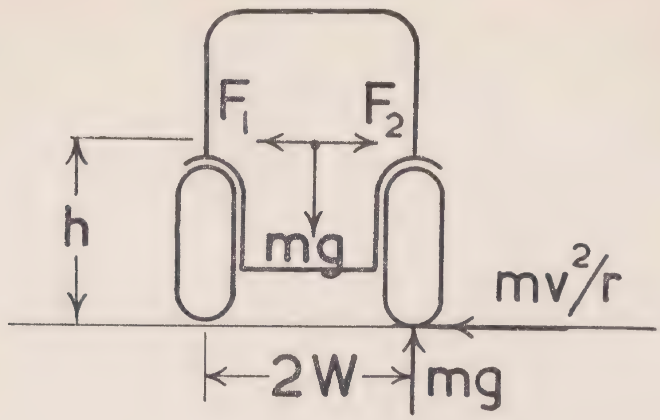


Fig. 4.31

to the weight vertically upwards through the point of contact of the outside wheel with the road, and also a horizontal frictional force equal to the centripetal force $\frac{mv^2}{r}$ acting inwards at this point of contact.

The frictional force acting at a distance h below the centre of gravity can be replaced by an equal force acting through the centre of gravity (F_1 in the diagram) and a second force F_2, which must be introduced to retain the equilibrium. With the frictional force this forms a couple $\frac{mv^2}{r} \, . \, h$.

The net effect on the car is thus:

(i) a force $F_1 = \frac{mv^2}{r}$ pushing it around the circle.

(ii) A couple $\frac{mv^2}{r} \, . \, h$ trying to overturn it.

(iii) A couple $mg \, . \, W$, where $2W$ is the width of the car, formed by the weight of the car and the reaction at the outer wheel. The couple acts so as to keep the car on an even keel.

If $\frac{mv^2h}{r} > mg\,W$, then the car will start to overbalance, and since any motion in this direction will increase the vertical distance h and decrease the horizontal distance W, the inequality will be made worse and the car is bound to go right over.

The maximum speed for safety in this case is given by:

$$\frac{mv^2_{\max}\,h}{r} = mgW,$$

$$\text{or } v_{\max} = \sqrt{\frac{grW}{h}} \qquad . \quad . \quad . \quad . \quad (56)$$

(*b*) *Car on a Banked Curve*

It is well known that a banked corner can be taken at a greater speed than a corner on a flat road. The reason for this is that the reaction R between vehicle and road has a horizontal component which can assist the frictional forces in supplying a much larger centripetal force. The forces acting in this

case (Fig. 4.32) are the friction F between wheels and road (note that F may be directed as shown in the diagram or in the opposite direction, depending on whether the vehicle is tending to skid outwards or to slide sideways down the slope), the normal reaction R of the road on the vehicle (R and F are shown applied to one wheel, in practice they would be distributed between all four), and the weight mg.

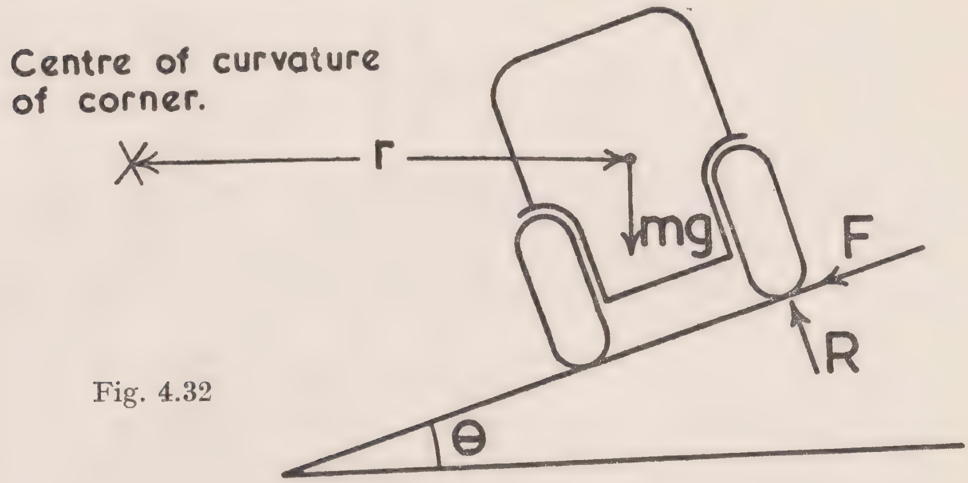


Fig. 4.32

If the car is to get around a bend of radius r at speed v, then the horizontal components of R and F must provide an inward force $\frac{mv^2}{r}$. Obviously the curve can be banked at such an angle that the whole of this force is supplied by the horizontal component of the reaction R alone, in this case there is no need for the frictional force F, i.e. the car can round the corner without skidding sideways even if the road is perfectly smooth.

If this is the case, then $F = 0$ and, resolving horizontally:

$$R \sin \theta = mv^2/r.$$

Resolving vertically gives: $R \cos \theta = mg$.

And eliminating R leads to: $\tan \theta = v^2/gr$ . . . . . (57)

This gives the best banking angle for vehicles which propose to take the bend at a speed v, or rewriting the expression as:

$$v = \sqrt{gr \tan \theta} \quad . \quad . \quad . \quad . \quad (58)$$

gives the best speed (on an icy day) at which to take a bend banked at the angle θ. Under good conditions, however, it is possible to take a corner at a much greater speed. In fact the speed can be increased until the force F parallel to the road surface becomes equal to limiting friction.

The horizontal components of R and F must still supply the centripetal force, thus referring again to Fig. 4.32:

$$R \sin \theta + F \cos \theta = mv^2/r.$$

But the maximum value that F can take is $\mu_s R$, where μ_s is the coefficient of static friction between wheels and road.

The maximum speed is given by:

$$\frac{mv^2_{\max}}{r} = R (\sin \theta + \mu_s \cos \theta),$$

but resolving vertically gives $R \cos \theta - F \sin \theta = mg$,

and substituting $\mu_s R$ for F, leads to:

$$R (\cos \theta - \mu_s \sin \theta) = mg.$$

Thus by eliminating R, we have:

$$\frac{mv^2_{\max}}{r} = mg\left[\frac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\right]$$

$$\text{or } v_{\max} = \left\{gr\left[\frac{\mu_s + \tan\theta}{1 - \mu_s\tan\theta}\right]\right\}^{\frac{1}{2}} \quad . \quad . \quad (59)$$

At a speed less than the optimum given by Equation (58), there will be a tendency for the vehicle to slip down the incline: in this case F will be reversed in direction, and there will be a minimum speed for rounding the bend to prevent slipping down. The working follows that given above and leads to:

$$v_{\min} = \left\{gr\left[\frac{(\tan\theta) - \mu_s}{1 + \mu_s\tan\theta}\right]\right\}^{\frac{1}{2}} \quad . \quad . \quad (60)$$

(iii) Aeroplane

When an aeroplane is in flight, the only *simple* forces acting on it are its weight and the 'lift' generated by the motion of the air past the wings: this lift acts perpendicularly to the surface of the wing. Apart from air resistance on the fuselage, there can be no sideways frictional force, hence the centripetal force necessary for the plane to make the turn must be provided in another way. This is done by the plane banking over at an angle θ (Fig. 4.33), so that the lift L acquires a horizontal component $L\sin\theta$: in this a plane is comparable with a car rounding a banked curve on a smooth road.

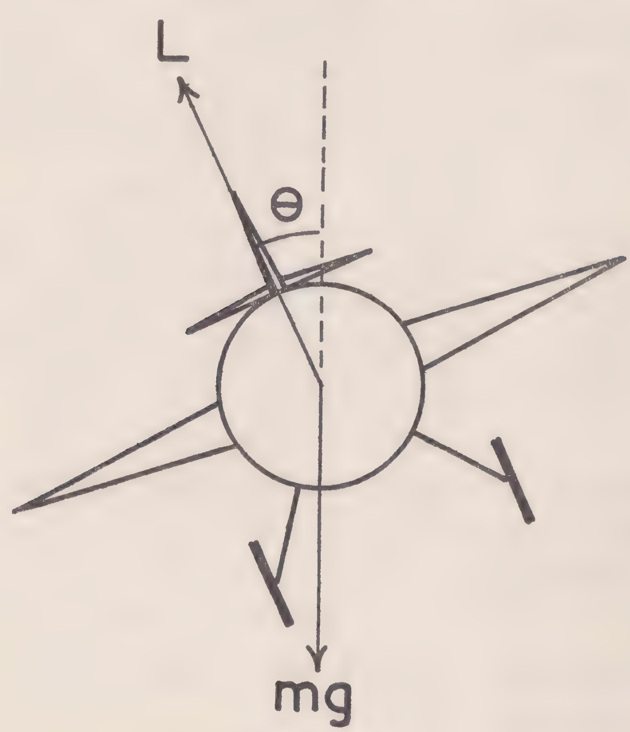


Fig. 4.33

The plane takes up a banked position so that:

$$L \sin \theta = \frac{mv^2}{r} \quad . \quad . \quad . \quad . \quad . \quad (61)$$

$$\text{and } L \cos \theta = mg,$$

$$\text{thus } \tan \theta = v^2/gr \text{ as before} \quad . \quad . \quad . \quad (62)$$

The plane is provided with ailerons to enable it to take up the banked position which is necessary for safe flight: this development was one of the main contributions of the Wright brothers.

4.22 Simple Harmonic Motion

If a point P moves around a circle with constant angular velocity ω (Fig. 4.34), and a perpendicular PN is drawn from P on to a diameter XX' of the circle, then N performs a particular motion along XX' as P goes around the circle. If N starts from O, it moves out to X, reverses in direction and moves back to O, on to X', where it reverses again, and thence in to O again. Thereupon N repeats this cycle as P goes around the circle for a second time and so on.

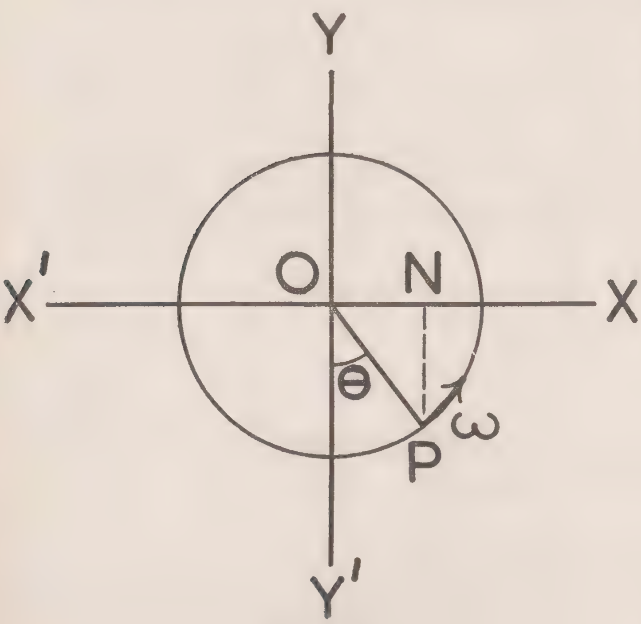


Fig. 4.34

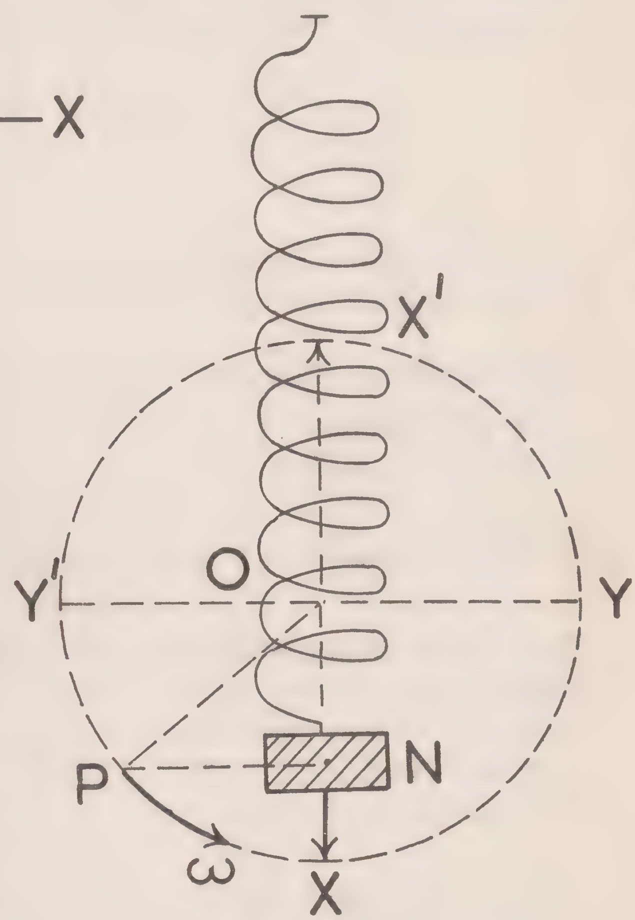


Fig. 4.35

The point N is said to perform *Simple Harmonic Motion*—it is characteristic of many vibratory or oscillatory systems.

First of all it is necessary to investigate various theoretical aspects of simple harmonic motion and then to apply these ideas to typical cases, but it should be remembered that this device of a point moving around a circle is only a convenient graphical way of develop-

ing the mathematics of such a familiar example as a weight bobbing up and down at the end of a stretched spring as shown in Fig. 4.35.

4.23 Amplitude, Periodic Time and Frequency

It will be noticed that the movement of N (Fig. 4.34) occurs about the centre O and varies between the limits X and X'. OX or OX' is called the *Amplitude* of the motion; for this the symbol A is normally used. It will be seen that the radius of the circle around which P moves is equal to A.

Since the motion of N is cyclic, the time taken for one cycle, called the *Periodic Time*, can be found. This is obviously equal to the time taken for P to go around the circle once. If P moves with an angular velocity ω, then it does this trip in $2\pi/\omega$ seconds,

$$\text{i.e. } \tau = \frac{2\pi}{\omega} \quad . \quad . \quad . \quad . \quad . \quad (63)$$

where τ is the periodic time.

Now if τ seconds is the time for 1 cycle, N will perform $1/\tau$ cycles per second, (cps), this is called the *Frequency* of the motion, symbol f.

$$\text{Thus } f = \frac{1}{\tau}$$

$$= \omega/2\pi$$

$$\text{or } \omega = 2\pi f. \quad . \quad . \quad . \quad . \quad (64)$$

ω is of course the angular velocity of P as it goes around the circle, but it is sometimes called the *Pulsance* of the motion of N.

4.24 Instantaneous Value of Displacement

The displacement of the point N measured from O at the instant shown in Fig. 4.34 is ON.

$$\text{But } ON = OP \sin \theta,$$

and OP is equal to A, the amplitude of the motion, thus:

$$ON = A \sin \theta.$$

Also, if time is measured from the instant when OP coincides with OY' (drawn perpendicular to XX'), then P moves from Y' to P in a time t seconds, and the angle θ is equal to ωt since P has an angular velocity ω. Substituting for θ in the equation above and writing x for the displacement ON, we have:

$$x = A \sin \omega t \quad . \quad . \quad . \quad . \quad (65)$$

This is the basic equation of a simple harmonic motion and if the equation for the displacement of any particle is similar to (65) then, straight away, its motion is known to be simple harmonic.

This equation allows x to take all the necessary values for N to perform a simple harmonic motion, for as t increases from zero, ωt (which is an angle), also grows from zero. Consequently $\sin \omega t$ takes all possible

values from 0 to $+ 1$ which it reaches when $\omega t = 90°$, from $+ 1$ back to 0 when $\omega t = 180°$, from 0 to $- 1$ when $\omega t = 270°$ and $- 1$ to 0 when $\omega t = 360°$. Thus x, which is equal to $A \sin \omega t$, can vary between $\pm A$ as ωt increases through the range 0–360°. As ωt goes from 360 to 720°, $\sin \omega t$ repeats the values that it took from 0 to 360°, and thus x goes through a second cycle and so on. From this it will be seen that the time for one cycle of the motion is the time taken for ωt to increase from 0 to 360° or 2π radians,

$$\text{thus } \omega\tau = 2\pi$$

$$\text{or } \tau = 2\pi/\omega \text{ as before.}$$

4.25 Phase

In the previous section, time was measured from the instant at which N passed through the centre of its motion moving towards X. If however it is decided to measure time from the instant when it goes through some other point such as N' moving towards O (Fig. 4.36), then this corresponds with measuring time from the instant when OP coincides with OP'.

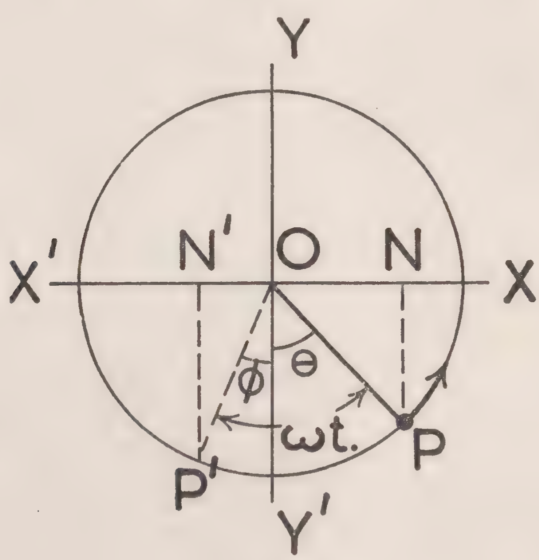


Fig. 4.36

The displacement ON is still given by $ON = A \sin \theta$, but θ is no longer equal to ωt. Instead:

$$\omega t = \theta + \varphi$$

$$\text{or } \theta = (\omega t - \varphi),$$

$$\text{thus } ON = A \sin (\omega t - \varphi) \quad . \quad . \quad . \quad (66)$$

The angle φ is called the *Initial Phase* of the motion (sometimes reduced to 'phase') and is determined entirely by the instant in the motion from which time is measured.

If we are concerned with two simple harmonic motions of the same frequency but occurring at different instants, then it is usual to start

measuring time as one of them passes through the point O, thus its equation is given by Equation (65), i.e.

$$x_1 = A_1 \sin \omega t.$$

The other motion will in general be passing through some other point at this instant, and its equation will be as given in Equation (66):

$$x_2 = A_2 \sin (\omega t - \varphi).$$

φ is then said to be the *Phase Difference* between the two motions; the phase of the second motion is said to be *lagging* on the first if φ is negative, *leading* if it is positive.

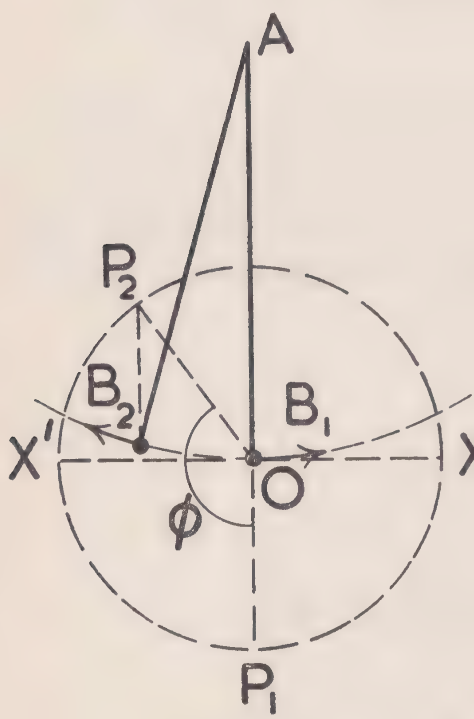


Fig. 4.37

For example, if two pendulums AB_1 and AB_2 of equal length are suspended from the same point A (Fig. 4.37), then the motion of the bobs is very nearly simple harmonic and of the same frequency. If at any instant the two bobs are moving as shown in the diagram and their motions have the same amplitude, then the phase difference between the two motions can be found by the following graphical construction.

The bobs have been illustrated at the instant when B_1 passes through the centre of its motion. With B_1 as centre, draw a circle of radius OX equal to the amplitude of the motion. Draw $X'X$ perpendicular to AB_1. From B_1 and B_2 draw lines perpendicular to XX' to cut the circle (top half if bob is moving to the left, bottom half if to the right) in P_1, P_2. Join P_1O (coincides with P_1B_1) and P_2O, then OP_1 and OP_2 are the rotating radii which can be used to generate the simple harmonic motions of the two bobs. The angle P_1OP_2 is thus the phase difference between the two motions.

However the two bobs may move subsequently, the radii OP_1 and OP_2 will rotate with the same angular velocity (since the pendulums have the same period), hence the angle P_1OP_2 will remain constant and there will be a constant phase difference between the two motions.

4.26 Instantaneous Value of Velocity

Since the point P moves around a circle with constant angular velocity ω, it everywhere has a tangential velocity given by $v = A\omega$, and the velocity of N at any point will be equal to the component parallel to XX' of the velocity of P (see Fig. 4.34).

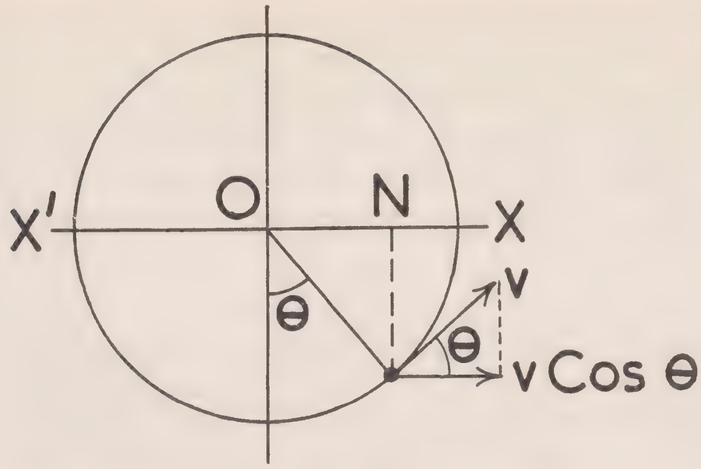


Fig. 4.38

From Fig. 4.38 it will be seen that this component is equal to $v \cos \theta$. Writing dx/dt for the velocity of N gives:

$$\frac{dx}{dt} = v \cos \theta,$$

$$= A\omega \cos \theta.$$

$$\text{but } \cos \theta = PN/PO = PN/A,$$

$$\text{thus } \frac{dx}{dt} = \omega PN,$$

$$= \pm \omega \sqrt{A^2 - x^2} \quad \text{since } ON = x$$

$$\text{or } \left(\frac{dx}{dt}\right)^2 = \omega^2(A^2 - x^2) \qquad (67)$$

This expression can also be obtained by differentiating (with respect to time) the expression for the displacement, thus, from Equation (65):

$$x = A \sin \omega t,$$

$$\text{hence } \frac{dx}{dt} = A\omega \cos \omega t,$$

$$\text{but } \cos \omega t = \pm \sqrt{1 - \sin^2 \omega t}.$$

$$\text{Thus } \frac{dx}{dt} = \pm A\omega \sqrt{1 - \sin^2 \omega t},$$

$$\text{also } \sin \omega t = x/A,$$

$$\text{giving } \frac{dx}{dt} = \pm A\omega \sqrt{1 - x^2/A^2}$$

$$\text{or } \left(\frac{dx}{dt}\right)^2 = \omega^2(A^2 - x^2) \text{ as before.}$$

The ambiguity of sign arises from the fact that N passes through every point in the motion twice per cycle with the same speed but going in opposite directions.

Equation (67) provides another method of recognising a simple harmonic motion. If it is found that the equation for the velocity of a particle at any instant is similar to Equation (67), then the motion of the particle is simple harmonic and the amplitude and periodic time can be read off from the equation. Such an equation usually arises when the motion of a system is investigated by means of its energy equation.

Example 4. *A particle of mass M is hung at the end of a light elastic cord obeying Hooke's Law (i.e. the tension in the string is proportional to its extension), and it is found that in the rest position the cord is stretched by an amount X. Investigate the motion of the particle if it is now depressed beyond its rest position and then released.*

Since the tension in the string is proportional to the extension, this may be expressed as:

$$T = kx,$$

where k is the constant of proportionality and x the extension of the string. At the rest position:

$$Mg = kX$$

$$\text{or } k = Mg/X.$$

If the particle is now depressed and released it will bob up and down so that when it is at a distance x below its original rest position it has a velocity v. An equation describing this motion can be obtained by writing down the expression for the total energy of the system and then equating this to some constant value since energy is conserved within the system.

The energy of the system when the particle is at a distance x below its original position is made up of the kinetic energy of the particle, the potential energy of the particle and the energy stored in the stretched cord, thus:

$$\text{Kinetic energy of particle} = \tfrac{1}{2}Mv^2.$$

$$\text{Potential energy of particle} = -Mgx.$$

The energy stored in a stretched cord is shown in Chapter 8 to be given by $\frac{1}{2}k$ (extension)^2, thus

$$\begin{aligned}
\text{Energy stored at rest position} &= \tfrac{1}{2}kX^2. \\
\text{Energy stored at depressed position} &= \tfrac{1}{2}k(X + x)^2. \\
\text{Gain in energy due to depression} &= \tfrac{1}{2}k(X + x)^2 - \tfrac{1}{2}kX^2 \\
&= kXx + \tfrac{1}{2}kx^2 \\
&= Mgx + \tfrac{1}{2}kx^2 \quad (\text{since } kX = Mg). \\
\text{Total Energy} &= \tfrac{1}{2}Mv^2 - Mgx + Mgx + \tfrac{1}{2}kx^2 \\
&= \tfrac{1}{2}(Mv^2 + kx^2).
\end{aligned}$$

If energy is to be conserved, this total energy must remain constant in value throughout the motion. The value of the constant can be found by considering the particle when passing through the normal rest position. If its velocity is then V, the only energy possessed by the system is the kinetic energy of the particle; this is equal to $\frac{1}{2}MV^2$, thus:

$$\begin{aligned}
\tfrac{1}{2}(Mv^2 + kx^2) &= \tfrac{1}{2}MV^2 \\
\text{or } v^2 &= \frac{1}{M}(MV^2 - kx^2) \\
&= \frac{k}{M}\left\{\frac{M}{k}V^2 - x^2\right\}.
\end{aligned}$$

Comparing this with Equation (67), i.e.:

$$\left(\frac{dx}{dt}\right)^2 = \omega^2(A^2 - x^2),$$

it is seen that the two are of the same form, hence the motion of the particle is simple harmonic about the normal rest position, further:

$$\omega^2 = \frac{k}{M}$$

$$\text{and } A^2 = \frac{MV^2}{k}$$

Thus the amplitude of the motion is given by:

$$A = \pm V\sqrt{\frac{M}{k}},$$

also, remembering that

$$\tau = 2\pi/\omega,$$

$$\tau = 2\pi\sqrt{\frac{M}{k}}.$$

4.27 Instantaneous Value of Acceleration

The acceleration of a point N performing a simple harmonic motion can be investigated by considering the motion of the point P. As this point moves around a circle of radius A (Fig. 4.39) with an angular

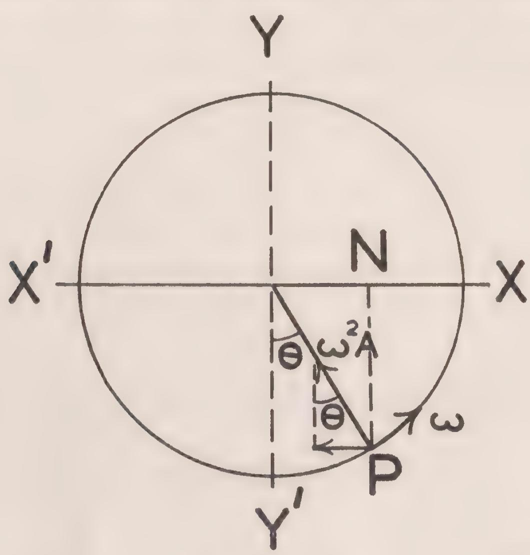


Fig. 4.39

velocity ω, it has a radial acceleration equal to $\omega^2 A$; the acceleration of N along XX' is thus equal to the component of the acceleration of P parallel to XX'. Writing $\frac{d^2x}{dt^2}$ for this acceleration gives:

$$\frac{d^2x}{dt^2} = -\omega^2 A \sin\theta.$$

(The negative sign is used to show that the acceleration is always directed back towards the origin in the opposite direction to the displacement.)

$$\text{But } x = A \sin \theta,$$

$$\text{thus } \frac{d^2x}{dt^2} = -\omega^2 x \quad . \quad . \quad . \quad . \quad (68)$$

This equation indicates that, in a simple harmonic motion, the acceleration is proportional to the displacement and directed towards the centre of the motion. If both sides of Equation (68) are multiplied by the mass of the particle, we get

$$\frac{md^2x}{dt^2} = -m\omega^2 x \quad . \quad . \quad . \quad . \quad (69)$$

and, remembering that $\dfrac{md^2x}{dt^2}$ is a force, we have the alternative statement that a particle performing a simple harmonic motion is subjected to a restoring force proportional to the displacement of the particle.

Equations (68) and (69) are perhaps the most important ones in the whole of this theory. These equations are the ones by which a simple harmonic motion is usually recognised.

Equation (68) can be manipulated to give an expression very suitable for calculating the period of a simple harmonic motion,

$$\text{thus } \omega^2 = \frac{d^2x/dt^2}{x}$$

(considering only the magnitude of the acceleration and displacement).

$$\text{But } \tau = 2\pi/\omega,$$

$$= 2\pi \sqrt{\frac{x}{d^2x/dt^2}},$$

$$= 2\pi \sqrt{\frac{1}{\dfrac{d^2x}{dt^2}\Big/ x}}.$$

Now $\left(\dfrac{d^2x}{dt^2}\Big/ x\right)$ is numerically equal to the acceleration occurring at a displacement of $x = 1$, thus this equation can be written as:

$$\text{Period} = 2\pi \sqrt{\frac{1}{\text{Acceleration at unit displacement}}} \quad . \quad (70)$$

Further, multiplying numerator and denominator inside the root sign by the mass of the particle gives:

$$\tau = 2\pi \sqrt{\frac{m}{m\dfrac{d^2x}{dt^2}\Big/ x}},$$

which can be written as:

$$\text{Period} = 2\pi \sqrt{\frac{\text{Mass}}{\text{Restoring force at unit displacement}}} \quad . \quad (71)$$

since md^2x/dt^2 is the restoring force acting on the particle.

Either of these expressions provides a quick method of finding the period of a system known to be executing a simple harmonic motion.

Example 5. *Particles of powder are scattered on the diaphragm of a telephone receiver mounted horizontally. The diaphragm executes a simple harmonic motion in the vertical direction at a frequency of* 1000 *cps. Find the maximum amplitude of the motion of the diaphragm if the particles remain in contact with it.*

If the particles are not fixed to the diaphragm, their downward motion is provided entirely by gravity. This can produce a maximum downward acceleration of 981 cm.sec^{-2}.

If the particles are to remain in contact with the diaphragm, then the maximum acceleration of the diaphragm must be less than 981 cm.sec^{-2}; if, however, the motion of the diaphragm is simple harmonic, it is described by the equation

$$\text{acceleration} = -\omega^2 x,$$

hence the maximum downward acceleration occurs at the maximum upward displacement, i.e. when the displacement is equal to the amplitude, A, of the motion.

For the particles to remain on the diaphragm:

$$\begin{aligned} \omega^2 A &\leqslant 981. \\ \text{But } \omega &= 2\pi f \\ &= 2 \,.\, \pi \,.\, 1000 \\ \therefore\ (2000\pi)^2 A &\leqslant 981 \\ \text{or } A &\leqslant \frac{981}{4\pi^2 \times 10^6}\ \text{cm}, \end{aligned}$$

so that the maximum amplitude is some 0·00025 mm.

4.28 Examples of Simple Harmonic Motion

(i) Mass Hung on an Elastic Cord

This case has already been examined on page 96, but the following alternative methods are interesting.

The mass M, hung on an elastic cord obeying Hooke's Law, will normally rest in equilibrium with the cord stretched to a length X where

$$Mg = kX.$$

If the particle is depressed a further distance x, the tension in the cord becomes $k(X + x)$, and the resultant upward force on the particle is $k(X + x) - Mg$, which reduces to kx since kX is equal to Mg.

We have called a depression $+ x$, so this *upward* force should be written as $- kx$, and remembering that force = mass × acceleration, the particle will move with an acceleration given by:

$$-kx = M \times \text{acceleration}$$

$$\text{or acceleration} = -\frac{k}{M} \,.\, x.$$

Since k/M is a constant, the acceleration is proportional to the displacement and directed back to the normal rest position, hence the

motion is simple harmonic. Comparing this equation with Equation (68), i.e.

$$\text{acceleration} = -\omega^2 x$$

$$\text{gives } \omega^2 = \frac{k}{M},$$

$$\text{but } \tau = 2\pi/\omega$$

$$= 2\pi\sqrt{\frac{M}{k}} \text{ as was found before.}$$

Alternatively, since acceleration is equal to $-\frac{k}{M}x$, then:

restoring acceleration at unit displacement

$$= k/M \text{ (writing } x = 1)$$

and from Equation (70):

$$\text{Period} = 2\pi\sqrt{\frac{1}{k/M}}$$

$$\text{or } \tau = 2\pi\sqrt{\frac{M}{k}}.$$

Again, since the restoring force is $-kx$, the restoring force at unit displacement is equal to k, therefore, from Equation (71), the period is given by:

$$\tau = 2\pi\sqrt{\frac{M}{k}}.$$

This device is sometimes used to find the gravitational acceleration g, for:

$$Mg = kX,$$

where X is the static displacement produced by the mass M, hence

$$g = \frac{k}{M} \cdot X.$$

But from the equation above

$$\frac{k}{M} = \frac{4\pi^2}{\tau^2},$$

$$\text{thus } g = \frac{4\pi^2 X}{\tau^2}.$$

The treatment given above assumes that the cord has no mass. It can be applied to the motion of a body on a spring, provided that a correction for the mass of the spring is applied.

An elastic cord can apply only a force of tension, thus if the amplitude of the motion is so large that the mass is carried above the normal unstretched position of the cord, the tension becomes zero and the

downward acceleration of the mass becomes constant (equal to g) so long as it is above this position. The acceleration is thus no longer proportional to the displacement and the motion ceases to be simple harmonic.

(ii) Column of Liquid Oscillating in a U-Tube

A liquid of density ϱ is contained in a U-tube of cross-sectional area A (Fig. 4.40). The total length of the column of liquid is l.

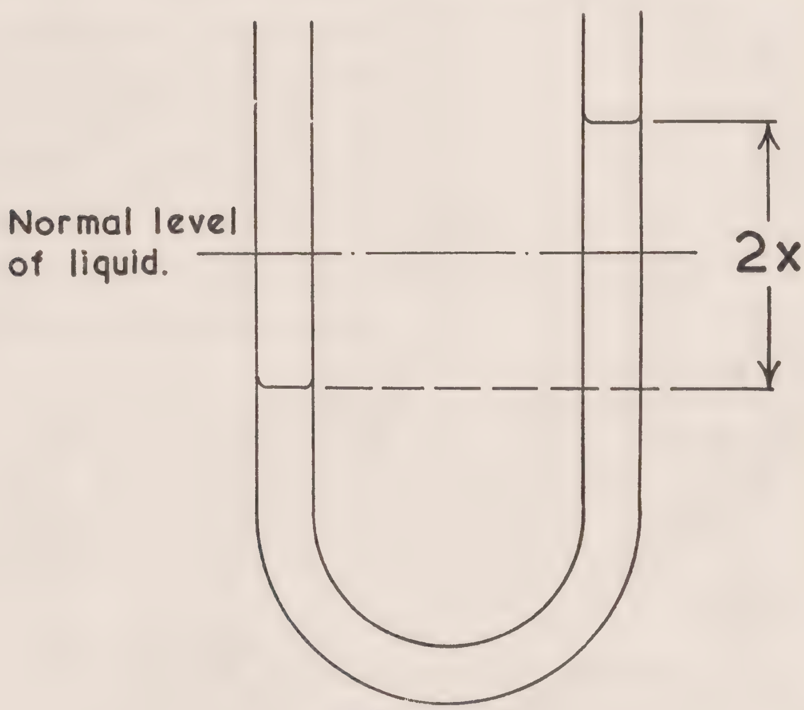


Fig. 4.40

Normally the fluid will set with the menisci in the two arms in the same horizontal plane; if, however, the liquid is disturbed from this position so that one meniscus rises a distance x, the excess weight of liquid in this limb will be $2xA\varrho g$. This is the restoring force acting on the column of liquid and it is seen to be proportional to the displacement x, hence the motion must be simple harmonic.

The restoring force at unit displacement is $2A\varrho g$, and since the total mass of the liquid column is $lA\varrho$, the period, as given by Equation (71), becomes

$$\tau = 2\pi\sqrt{\frac{lA\varrho}{2A\varrho g}},$$

$$= 2\pi\sqrt{\frac{l}{2g}} \quad . \quad . \quad . \quad . \quad . \quad (72)$$

(iii) Simple Pendulum

A simple pendulum consists of a heavy but very small bob of mass m attached to a light, flexible, but inelastic cord of length l; this cord, at

its upper end, is fixed to a rigid support A (Fig. 4.41). If the bob is pulled aside from its rest position and then released, it will oscillate along an arc of the circle with centre at A.

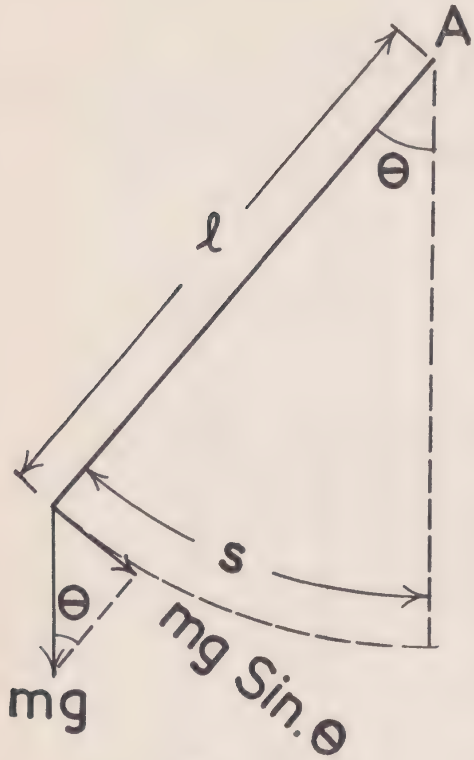


Fig. 4.41

If the bob is at a distance s, measured along the arc, from its rest position, then the string will make an angle θ with the vertical where:

$$s = \theta l. \quad (\theta \text{ measured in radians.})$$

The forces acting on the bob when at this position are the tension in the string and the weight of the bob, but of these only the component of the latter perpendicular to the string can have any restoring effect.

From the diagram it is seen that:

$$\begin{aligned}\text{Restoring force} &= mg \sin \theta \\ &= mg \sin(s/l).\end{aligned}$$

It appears at first sight that the motion of the bob along the arc is not simple harmonic since the restoring force is not proportional to the displacement s but to $\sin(s/l)$.

The motion becomes approximately simple harmonic, however, if the arc through which the bob swings is very small. If this is so, then θ (measured in radians) is small and $\sin \theta$ is approximately equal to θ, thus:

$$\text{Restoring force} \simeq mg\,\theta$$

$$\text{or restoring force} \simeq \left(\frac{mg}{l}\right) s \quad . \quad . \quad . \quad . \quad . \quad (73)$$

The restoring force is thus proportional to the displacement and the motion is simple harmonic as long as θ is sufficiently small for the approximation noted above to be valid.

Equation (73) leads to:

$$\text{Restoring force at unit displacement} = \frac{mg}{l}$$

and from Equation (71) the period of the pendulum is given by:

$$\begin{aligned}\tau &= 2\pi\sqrt{\frac{m}{mg/l}}, \\ &= 2\pi\sqrt{\frac{l}{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)\end{aligned}$$

In practice a simple pendulum can swing through an angle of nearly

24° on either side of the vertical before this approximation for the period is incorrect by as much as 1 per cent.

Example 6. *The period of a simple pendulum is 1·6 seconds and the mass of the bob is 60 gm. The bob is pulled aside through an arc distance of 8 cm and then released. Find its kinetic energy and displacement after 0·6 sec.*

For small displacements, the motion of the bob of a simple pendulum along the arc is simple harmonic, thus its motion may be represented by

$$S = S_{\max} \sin\left\{\frac{2\pi t}{\tau} + \varphi\right\}$$

where:

S is the arc displacement,
$S_{\max}$ is the amplitude of the motion,
τ is the period of the motion,
φ is the initial phase of the motion.

In this case $S_{\max} = 8$ cm,
and $\tau = 1{\cdot}6$ sec,

$$\text{thus } S = 8 \sin\left\{\frac{2\pi t}{1{\cdot}6} + \varphi\right\}.$$

Also, if time is measured from the instant at which the bob is released, $t = 0$ when $S = 8$ cm.

Hence $\sin\varphi = 1$
or $\varphi = 90°$,

$$\text{giving } S = 8 \sin\left\{\frac{2\pi t}{1{\cdot}6} + 90°\right\}$$

$$= 8 \cos\frac{2\pi t}{1{\cdot}6}.$$

Thus, when $t = 0{\cdot}6$ sec,

$$S = 8 \cos\frac{2\pi \,.\, 0{\cdot}6}{1{\cdot}6}$$

$$= 8 \cos 135°$$

$$= -\,5{\cdot}66 \text{ cm (i.e. } 5{\cdot}66 \text{ cm on the other side}$$

of the rest position).

The velocity of the bob at this point is given by

$$v^2 = \omega^2\,(8^2 - 5{\cdot}66^2)$$

also $\omega = \dfrac{2\pi}{1{\cdot}6}$, and the kinetic energy is given by $\frac{1}{2}mv^2$, therefore:

$$\text{kinetic energy} = \tfrac{1}{2} \times 60 \times \left(\frac{2\pi}{1{\cdot}6}\right)^2 (8^2 - 5{\cdot}66^2)$$

$$= 30 \times \left(\frac{\pi}{0{\cdot}8}\right)^2 (64 - 32)$$

$$= 30 \times 15{\cdot}4 \times 32$$

$$\doteq 14{,}800 \text{ ergs.}$$

4.29 Rotational Simple Harmonic Motion

So far we have only considered the simple harmonic motion of bodies moving in a straight line or a very small arc of a curve, but it is also possible for a body performing a pure rotation to undergo a simple harmonic motion. The motion of the balance wheel of a watch under

the action of the spiral hair-spring is an example of such a rotationa simple harmonic motion. It is analogous to the linear simple harmonic motion performed by a mass oscillating vertically at the end of an elastic cord, if the angular displacement of the wheel from rest is considered instead of the linear displacement of the mass.

If the rotational motion of a body is such that its angular acceleration is proportional to its angular displacement from some position and tends to return it to that position, then the rotational motion is described as a simple harmonic motion. As in the linear case (see page 98), this can be extended as follows: the angular motion of a body is simple harmonic if the restoring couple is proportional to the displacement. The equations developed for linear motion can all be adapted for rotational motion, and in particular the angular counterparts of Equations (70) and (71) become:

$$\text{Period} = 2\pi\sqrt{\frac{1}{\text{Angular Acceleration at Unit Angular Displacement}}} \quad . \quad (75)$$

$$\text{or Period} = 2\pi\sqrt{\frac{\text{Moment of Inertia}}{\text{Restoring Couple at Unit Angular Displacement}}} \quad . \quad . \quad (76)$$

4.30 Examples of Rotational Simple Harmonic Motion

(i) The Simple Pendulum

The simple pendulum can be considered as a case of rotation of the whole pendulum about the point of suspension A in Fig. 4.41.

When the pendulum is displaced through an angle θ, the restoring force on the bob is $mg \sin \theta$ and the couple exerted by this force about the point of suspension A is $mgl \sin \theta$. Now if the displacement is small, θ can be written for $\sin \theta$, whence:

$$\text{Restoring Couple} \simeq mgl\theta.$$

Thus the restoring couple is proportional to the angular displacement θ and the rotational motion is simple harmonic.

The moment of inertia of the bob about A can be found from the parallel axes theorem. If the bob is a sphere, its moment of inertia about an axis through its centre is $\frac{2}{5}mr^2$ where r is the radius of the bob, thus about A the moment of inertia is $\{\frac{2}{5}mr^2 + ml^2\}$. Normally $\frac{2}{5}mr^2$ is much smaller than ml^2 and is ignored (this constitutes one of the inherent errors in the theory of the simple pendulum, see page 142), thus the moment of inertia of the bob about the point of suspension is ml^2.

The period can now be found from Equation (76), giving:

$$\tau = 2\pi\sqrt{\frac{ml^2}{mgl}}$$

(since the restoring couple at unit displacement is mgl).

$$\text{Therefore } \tau = 2\pi\sqrt{\frac{l}{g}} \text{ as before.}$$

(ii) The Bifilar Pendulum

This pendulum consists of a lath AB of mass m supported by two vertical strings of length l; these are attached symmetrically to the lath at distances a from its centre of gravity (Fig. 4.42).

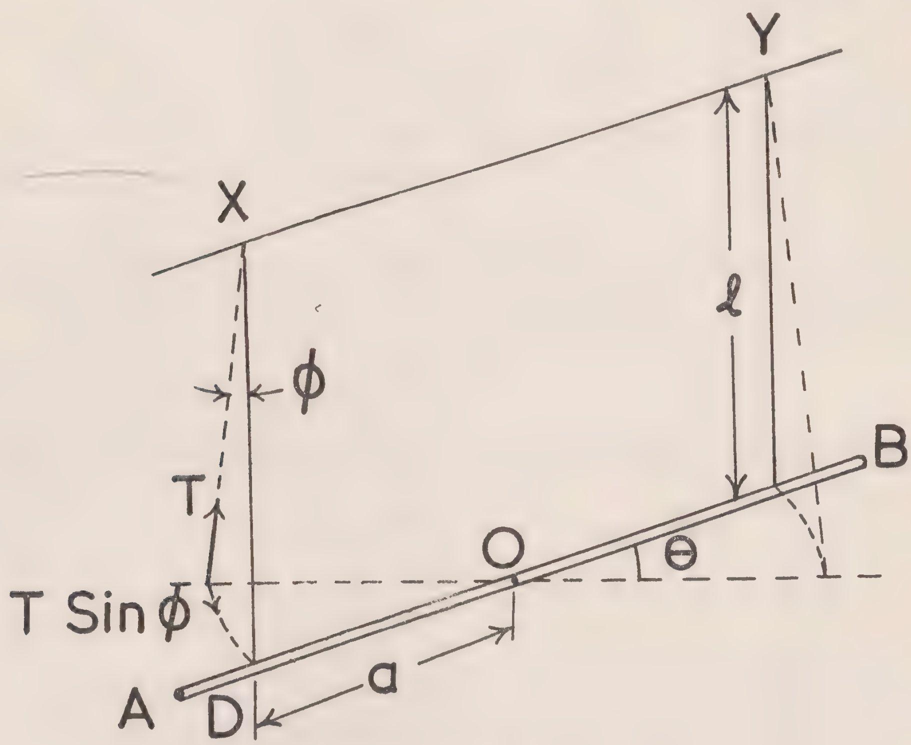


Fig. 4.42

If the lath is turned in the horizontal plane through a small angle θ, then the strings will turn through an angle φ where θ and φ can be related as follows; the path traced out by the point D can be considered as an arc of a circle centre O, whence arc $= a\theta$. It can also be considered as an arc of a circle centre X, and thus is also equal to $l\varphi$.

$$\text{Thus } a\theta = l\varphi.$$

If the tension in each string is T, then the component perpendicular to the lath is $T \sin \varphi$, and the restoring couple acting on it is given by:

$$\text{Restoring Couple} = 2aT \sin \varphi.$$

Now if a and l are of the same order of magnitude, both θ and φ can be restricted to be small; if this is so, then φ can be written for $\sin \varphi$ and:

$$\begin{aligned}\text{Restoring Couple} &\simeq 2aT\varphi \\ &= 2aT \cdot \frac{a\theta}{l} \\ &= \frac{2a^2T}{l}\theta.\end{aligned}$$

The restoring couple is thus proportional to the angular displacement and the motion of the lath is simple harmonic.

To find the period of the motion, it is necessary to evaluate T; this is done by resolving vertically, giving:

$$2T \cos \varphi = mg,$$

but if φ is small, $\cos \varphi = 1$, and $2T = mg$. Substituting this in the equation above gives:

$$\text{Restoring Couple} = \frac{a^2mg}{l} \cdot \theta$$

and restoring couple at unit angular displacement

$$= \frac{a^2mg}{l}.$$

If the moment of inertia of the lath about its centre of gravity is I, then from Equation (76) the period is seen to be:

$$\tau = 2\pi\sqrt{\frac{Il}{a^2mg}} \quad . \quad . \quad . \quad (77)$$

If we write $I = mk^2$ where k is the radius of gyration of the rod, then

$$\tau = 2\pi\sqrt{\frac{mk^2l}{a^2mg}}$$

$$= 2\pi\frac{k}{a}\sqrt{\frac{l}{g}} \quad . \quad . \quad . \quad . \quad (78)$$

The bifilar suspension provides a useful method of measuring the moment of inertia (about the centroid) of irregularly shaped objects—a shell, for instance, can be supported by two strings as in Fig. 4.43, and its period of oscillation measured; all the other factors appearing in Equation (77) can be measured directly, and thus the moment of inertia of the shell about the axis AB can be calculated.

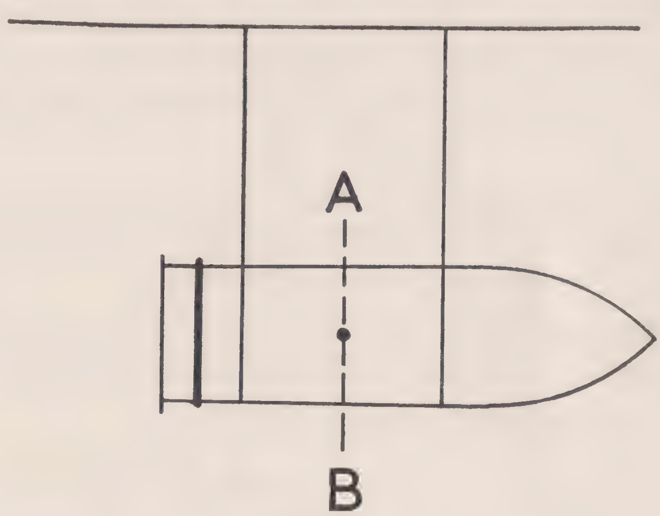


Fig. 4.43

Other examples of rotational simple harmonic motion to be considered later in this book are the compound pendulum (page 143) and the torsional pendulum (page 221).

SUMMARY OF NEW UNITS INTRODUCED IN THIS CHAPTER

Quantity	*Dimensions*	*c.g.s. and derived units*	*f.p.s. and derived units*	*Gravitational unit*
Moment / Couple / Torque	MLT^{-2}	gm.cm.sec^{-2} or dyne.cm	lb.ft.sec^{-2}	gm-wt.cm or lb-wt.ft
Angular Velocity	T^{-1}	radian.sec^{-1}	radian.sec^{-1}	
Angular Acceleration	T^{-2}	radian.sec^{-2}	radian.sec^{-2}	—
Moment of Inertia	ML^2	gm.cm^2	lb.ft^2	—
Radius of Gyration	L	cm	ft	—
Angular Momentum	ML^2T^{-1}	gm.cm^2.sec^{-1}	lb.ft^2.sec^{-1}	—
Frequency	T^{-1}	cycles.sec^{-1} cps	cycles.sec^{-1} cps	—

EXERCISES 4

1. State Newton's laws of motion, and explain the significance of the forces of 'action' and 'reaction' which appear in the third law by identifying them in a number of dynamic and static systems.

   Describe the effects that would be observed on the earth if in the course of one hour the angular velocity of the earth decreased uniformly from its normal value to zero. (Radius of earth 6300 km.)
   (Cambridge Univ. Schol., King's College Group.)

2. What is meant by (*a*) the centre of mass of a body, (*b*) the centre of gravity of a body?

   A cylindrical can is made of a material weighing 10 gm./sq.cm. and has no lid. The diameter of the can is 25 cm. and its height 50 cm. Find the position of the centre of mass when the can is half full of water. (Cambridge H.S.C.)

3. Obtain a formula for the position of the centre of gravity of a sector of a circle in terms of the angle subtended by the arc at the centre of the circle (2α) and the radius (r). In particular find the centre of gravity of a quadrant of a circle. (Manchester Univ. Schol.)

4. Discuss shortly the factors upon which the sensitivity of a balance depend.

   The density of an aqueous solution is measured using a specific gravity bottle and brass weights. At what precision does the effect of the buoyancy of the air become appreciable?
   (Manchester Univ. Schol.)

5. Describe and give the theory of a sensitive beam balance. State the factors which determine its sensitivity.

   What is meant by the method of 'double weighing'? What are its advantages? (London Univ. Inter. B.Sc.)

6. Give the theory of a simple beam balance, and specify the conditions that must be fulfilled for it to be (*a*) accurate, (*b*) sensitive.

The arms of a balance are each 15·0 cm. long, the mass of the beam is 200 gm., the centre of gravity of the beam is 1 mm. below the central knife-edge, and the three knife-edges are coplanar. A closed vessel of volume 300 c.c. is counterpoised when the atmospheric pressure is 76·5 cm. of mercury. Estimate the deflexion of the beam that is observed if the barometer falls to 75·0 cm., the temperature being unchanged.

(Take the density of air as 0·00125 gm. per c.c. under the initial conditions; the effect of air buoyancy on the counterpoising metal weights is negligible.) (Oxford G.C.E. Advanced level.)

7. Give the theory of the ordinary beam balance. Show how the sensitivity depends on the length of the arms, the mass of the beam, and the position of the centre of gravity of the beam.

A 300 c.c. beaker is placed on one pan of a balance and counterpoised. Carbon dioxide is then passed into the beaker until all the air has been displaced by it; find the additional weight that must be added in order to restore balance. After a little time it is found that an additional weight of 0·152 gm. suffices; find the proportion by volume of carbon dioxide remaining in the beaker at this stage.

(Take the density of air to be 1·26 gm. per litre, and that of carbon dioxide to be 1·96 gm. per litre.) (Oxford G.C.E. Advanced level.)

8. Show that any system of coplanar forces acting on a body can be replaced by a force whose line of action passes through any chosen point in the plane, together with a couple.

The arms of an ordinary beam balance are each 12 cm. long, the central knife-edge is 0·5 mm. above the line joining the two scale-pan knife-edges, while the centre of gravity of the beam is 1·5 mm. below this line. The mass of the beam is 250 gm., and that of each scale-pan is 80 gm. Find the deflexion of the beam from the horizontal when the loads placed in the two pans are 20·00 and 20·08 gm. respectively. (Oxford H.S.C.)

9. Explain the following facts:

(*a*) When a car is starting, the front is observed to jerk upwards,

(*b*) When the brakes are applied suddenly, the rear wheels are apt to lose their grip on the road.

The centre of mass of a car weighing 1,000 kg. lies symmetrically between the four wheels, at a height of 60 cm. above the road. The distance between the front and back axles is 3 metres. Calculate the normal reactions exerted on the ground by the front and back wheels while a constant force on the brakes is reducing the speed from 50 kilometres per hour to zero in five seconds. (Oxford Univ. Schol.)

10. Describe how you would determine the moment of inertia of a fly-wheel about its usual axis of rotation, and explain how the result is calculated.

Masses of 120 gm. and 100 gm. are suspended one at each end of a light cord passing over a pulley which is free to rotate about a fixed horizontal axis. The radius of the pulley is 5 cm. and its moment of inertia about its axis is 4,300 gm.cm.^{2} Assuming that the string does not slip round the pulley, find the linear acceleration of the masses. (Oxford H.S.C.)

11. Calculate the moment of inertia of a uniform circular disc about its axis.

A light string passes over a pulley of moment of inertia I and radius r. Masses m_1 and m_2 ($m_1 > m_2$) are attached to the ends of the string and allowed to move under the influence of gravity. Assuming no slip between the string and pulley, calculate the distance moved by the masses in time t. (Oxford Univ. Schol.)

12. Two gear wheels, of equal thickness, of the same material and having radii in the ratio 2 : 1, are mounted on parallel frictionless spindles, but are separated so as not to mesh with one another. The larger wheel is set spinning at a speed of 10 revolutions per second, and the wheels are then brought into mesh. What is the resulting speed of each wheel? (Cambridge Univ. Schol., King's College Group (Part).)

13. Explain what is meant by the *moment of inertia* of a body about a given axis, and deduce that of a uniform circular disc (*a*) about its axis of symmetry, (*b*) about a diameter.

Calculate the acceleration with which a penny will roll down the line of greatest slope of a rough plane inclined at an angle of 30° to the horizontal. (Cambridge Univ. Schol., King's College Group.)

14. A four-stroke, single-cylinder petrol engine has a light piston and crank shaft, which drive a flywheel of moment of inertia 10 kgm.cm.^{2} Each cycle of the engine comprises four strokes of the piston and includes the following operations:

(*a*) 125 c.c. of air and petrol vapour at atmospheric pressure (10^6 dynes cm.$^{-2}$) are drawn into the cylinder.

(*b*) The momentum of the flywheel forces the piston back and compresses the gases until their volume is 27 c.c.

(*c*) The mixture is exploded by an electric spark and the expanding gases drive the piston and accelerate the flywheel.

(*d*) The exhaust gases are expelled.

Find the lowest speed, in cycles per minute, at which the engine will operate, assuming that the mean angular velocity of the flywheel is 3/4 of its velocity at the beginning of the second stroke and that the ratio of the two specific heats of the gases is 4/3. (Manchester Univ. Schol.)

15. A garden roller consists of a uniform solid cylinder with a handle of negligible mass attached to a thin frictionless axle concentric with the cylinder. Show that the roller will always roll without slipping when pulled with constant *velocity* up any incline with the handle at any angle to the horizontal.

If the coefficient of friction between the roller and the ground is μ, find at what *acceleration* the roller begins to slip when pulled up

an incline making an angle θ with the horizontal if the handle is maintained parallel to the surface of the incline.

(Oxford Univ. Schol.)

16. A watch of moment of inertia 50 $gm.cm.^2$ contains a balance wheel of moment of inertia 0·5 $gm.cm.^2$ and goes correctly when the case is firmly held. Find how much it will gain if it is suspended, dial up, by a torsionless thread. (Manchester Univ. Schol.)

17. Define the moment of inertia of a body about a given axis, and obtain from first principles an expression for the kinetic energy of a rigid body rotating about a fixed axis with a given angular velocity.

A small flywheel has an axle of diameter 1 cm.; the mass of the whole body is 2000 gm., and its moment of inertia about an axis passing through the centre of the axle is 24,000 $gm.cm.^2$ The flywheel is placed with its axle on two parallel rails, which form a slope inclined to the horizontal, and it travels a distance of 400 cm. down this slope in 14 sec., starting from rest. Find the inclination of the rails to the horizontal, and also the kinetic energy of the flywheel at the end of this time. (Oxford G.C.E. Advanced level.)

18. Find, in foot-pounds, the kinetic energy of a uniform solid circular cylinder of 3 ft. radius weighing 1 ton when it is rotating at 120 revolutions per minute about a horizontal axis through its axis of symmetry.

If a tangential force of 5 lb. wt. is applied at the rim, find the number of revolutions the cylinder makes before coming to rest.

(London Univ. Inter. B.Sc.)

19. Prove that the moment of inertia of a uniform circular cylinder of mass M and radius r about its axis is $\frac{1}{2}Mr^2$.

A string, 8 feet long, is wrapped tightly round the axle of a wheel and is tied to a 60-lb. weight which falls under gravity until all the string is unwound. The wheel, of uniform material, weighs 120 lb. and its diameter is 14 inches. If the diameter of the axle is 3 inches and its moment of inertia is negligible, determine the angular velocity acquired by the wheel. (London Univ. Inter. B.Sc.)

20. Define *moment of inertia.*

Describe and explain any experiment you have seen or performed for determining a moment of inertia.

A cylinder, radius 5 cm., is mounted so that it can rotate, without friction, about a horizontal axis which coincides with its own axis. A light cord is wrapped twice round the cylinder, one end being looped over a small peg on the curved surface of the cylinder, the other supporting a mass of 100 gm. hanging freely. The cylinder and mass are held at rest and then released so that the cord unwinds and becomes detached after the cylinder has made two revolutions. If the moment of inertia of the cylinder about its axis is $2 \cdot 0 \times 10^4$ c.g.s. units, calculate (*a*) the angular velocity of the cylinder when the cord falls off, (*b*) the time taken from the instant of release of the mass until the cord becomes detached. (Northern Univ. H.S.C.)

21. Calculate the moment of inertia of a uniform circular disc about its axis.

A pulley in the form of a uniform circular disc of diameter 1 metre is mounted on a rough horizontal axle of diameter 5 cm., and is driven at a speed of 240 revolutions per minute. It is found that when the belt slips off, the pulley comes to rest after 1 minute. Assuming that the laws of sliding friction apply between the pulley and the axle, calculate the coefficient of friction. (Oxford Univ. Schol.)

22. Describe how you would determine the moment of inertia of a flywheel about its axis of rotation.

A flywheel of moment of inertia 10^5 gm.cm.^{2} is mounted on a horizontal axle 2 cm. in diameter. The frictional couple opposing its rotation is 2×10^4 dyne-cm. A mass of 500 gm. is attached to a fine thread wound round the axle, and hangs vertically. Assuming that the thread does not slip on the axle, find the time taken for this mass to fall from rest through a distance of 120 cm. Find also the kinetic energy of the flywheel at the end of this time.

(Oxford G.C.E. Advanced level.)

23. State Newton's law of elastic collision.

A ball of mass M rolls (along the line joining their centres) towards another ball which is at rest and of mass m. Derive an expression for the energy of the second ball after collision. Find the ratio of M to m in order that the fraction of the energy transferred may be a maximum.

How would you show experimentally that the coefficient of restitution of two bodies is independent of their relative velocity of impact?

(Cambridge Univ. Schol., Newnham and Girton Colleges.)

24. Show that a particle of mass m gm. describing a circle of radius r cm. with uniform angular velocity ω radians per second experiences a force $mr\omega^2$ dynes towards the centre of the circle.

A small ball of mass 500 gm. suspended from a fixed point by a light inextensible string 200 cm. long describes a horizontal circle of radius 100 cm., the string sweeping out a conical surface. Find the frequency of revolution of the ball, and the tension in the string.

(Oxford G.C.E. Advanced level.)

25. Find expressions for the tangential and normal components of the acceleration of a particle moving in a plane curve.

A cart is being driven along a country lane and lumps of mud are being flung off the rims of the wheels, which are of radius r. Find the minimum speed necessary in order that pieces of mud will rise higher than the tops of the wheels. If the cart travels at a speed v which exceeds this limit, find an expression for the maximum height that can be reached by a piece of mud flung off one of the rims.

(Cambridge Univ. Schol., King's College Group.)

26. Define coefficient of friction, and distinguish between static and kinetic coefficients. Explain how you would determine the static coefficient between two surfaces by an inclined plane method, proving the formula you use.

A sixpence is placed on a gramophone turntable which, when set

rotating, reaches a maximum speed of 78 r.p.m. When the centre of the coin is not more than 10 cm. from the centre of the turntable, the coin remains on the table; at a greater distance, it slides off. Explain this, and calculate the coefficient of friction between the coin and turntable. (Cambridge G.C.E. Advanced level.)

27. Explain Newton's third law of motion and use it to show that linear momentum is conserved when a system of particles is subject to external forces.

Calculate at what angle a skater leans inwards when moving in a circle of 50-ft. radius with a velocity of 15 ft. per second.

(Manchester Univ. Schol.)

28. *Criticise* briefly the following statement and *amend* it where necessary:

A particle moving in a circle with constant speed v experiences a force outwards along the radius equal to mv^2/r. More work is therefore required to drive the particle in this circle than would be necessary to drive it with velocity v in a straight line.

(Cambridge Univ. Schol., King's College Group (Part).)

29. Obtain an expression for the tension in a hoop of radius r, mass M, revolving in a horizontal plane with angular velocity ω.

Find the rim velocity at which a solid rubber tyre of mass 20 lb. will become slack on a wheel of 24-in. diameter, if the tension in the tyre at rest is 800 lb.wt. (Oxford Univ. Schol.)

30. Define Simple Harmonic Motion, and from your definition establish the linear relation between the square of the velocity and the square of the displacement from the mean position.

Calculate the amplitude and the periodic time for such a motion if the velocity is 48 ft. per sec. when the displacement from the mean position is 7 ft., and 40 ft. per sec. when the displacement is 15 ft.

(London Univ. Inter. B.Sc.)

31. A particle of mass 2 lb. is executing simple harmonic motion under the action of a variable force F. The amplitude is 5 ft. and the period 4 sec. Find the maximum speed of the particle and the rate at which the force F is working when the particle is 3 ft. from the centre of oscillation. Find also its maximum rate of working.

(London Univ. Inter. B.Sc.)

32. Show how the kinetic energy of a body executing linear simple harmonic motion varies with the displacement during one-quarter of a period.

A shelf on which rests a weight of 1 lb. oscillates vertically with S.H.M. and its velocity has the values 2 ft. per sec. and 1 ft. per sec. when its distances from the mean position are 1 ft. and 2 ft. respectively. Find the period of the motion and the maximum and minimum forces exerted by the weight on the shelf.

(Manchester Univ. Schol.)

33. What are the conditions for producing Simple Harmonic Motion? Give examples of S.H.M. drawn from as wide a range of physics as possible, making clear what quantity it is that varies and why its variations resemble a S.H.M.

A small body rests on the horizontal diaphragm of a loudspeaker which is being supplied with an alternating current of constant amplitude but the frequency of which can be varied. If the amplitude of the movement of the diaphragm is 10^{-3} cm. at all frequencies, find the greatest frequency for which the small body stays in contact with the diaphragm. (Cambridge Univ. Schol., King's College Group.)

34. Explain carefully what is meant by simple harmonic motion and give examples of it from as many different branches of physics as you can.

A mass is attached to the bottom end of an unstretched light spring whose upper end is fixed. The mass is suddenly released and is observed to oscillate between extreme positions 20 cm. apart. Assuming that the extension is proportional to the tension, calculate the period of oscillation. (Cambridge Univ. Schol., King's College Group.)

35. What is meant by simple harmonic motion? Describe *two* experiments which show that the motion of a mass oscillating at the end of a vertical spring satisfies the conditions of simple harmonic motion.

The upper end of a light vertical spring is fixed and on the lower end is a scale pan of mass 20 gm. A mass of 20 gm. is introduced into the scale pan. Prove that the resulting vertical motion of the scale pan and weight is simple harmonic, and calculate the period and amplitude if the spring constant is 10 gm. weight for 1 cm. extension. (Cambridge G.C.E. Advanced level.)

36. Explain what is meant by simple harmonic motion and show that the vertical oscillations of a mass fixed to the lower end of a light spring are simple harmonic, provided that Hooke's law is obeyed.

A particle of mass 300 gm. is fixed to the lower end of such a spring, and produces a steady displacement of 5 cm. It is pulled down through a further 5 cm. and is then released. Find the period of the motion, and also the kinetic energy with which the mass passes through the mean position of the motion.

(Oxford G.C.E. Advanced level.)

37. Show that small vertical oscillations of a mass suspended by a light spring from a rigid support are simple harmonic. What condition must be fulfilled if the motion is to remain simple harmonic when the amplitude is no longer small? Why do the oscillations gradually decrease in amplitude as the mass continues to oscillate?

Describe an experiment in which a loaded spring is used to determine the acceleration due to gravity.

A spring is such that a load of 100 gm. stretches it by 20 cm. When a load of 60 gm. is attached to the spring and set oscillating vertically there are 50 oscillations in 34·7 seconds. Calculate the acceleration due to gravity.

(Northern Univ. G.C.E. Advanced level.)

38. Describe experiments you would carry out with a helical spring to determine the acceleration due to gravity and give the theory underlying the calculation.

When a load of 100 gm. wt. is gradually applied at the lower end of a light helical spring, whose upper end is clamped, the lower end

descends through 10 cm. Supposing that the load is suddenly applied to the unstretched spring, calculate (*a*) the maximum extension of the spring, (*b*) the strain energy stored in the spring in that position, (*c*) the kinetic energy of the 100-gm. mass as it passes through the equilibrium position. (Northern Univ. G.C.E. Schol. level.)

39. Prove that the period of a simple pendulum of length l is $2\pi\sqrt{(l/g)}$.

A mine-shaft lift descends 200 ft. from rest with uniform acceleration in 20 sec. If a clock with a seconds pendulum (i.e. having a period of 2 sec.) is carried in the lift, find at what rate, in seconds per hour, the clock will gain or lose. (London Univ. Inter. B.Sc.)

40. Describe in detail how you would determine the acceleration due to gravity, using a very long simple pendulum suspended from an inaccessible point. Show how you would obtain the result from your observations.

The pendulum of a clock consists of a very thin rigid steel rod to the lower end of which is fixed a massive steel bob. If the clock keeps correct time at 15° C., estimate its error, due to expansion of the pendulum, in a 24-hour period when the temperature rises to 25° C. The coefficient of linear expansion of steel may be taken as 12×10^{-6} deg^{-1}. C. (Northern Univ. H.S.C.)

41. A uniform rigid rod of length 3 ft. and mass 2 lb. has a small ball of mass 4 lb. fixed to one end. The other end is hinged so that the rod can move freely in a vertical plane. The rod is held in a horizontal position by attaching it to the lower end of a vertical helical spring at a point 1 ft. from the hinge. A force of 10 lb. weight will stretch the spring 1 ft. The system is made to oscillate by slightly depressing the ball from its equilibrium position. Show that the motion of the ball is simple harmonic and calculate the period of vibration. What is the maximum kinetic energy of the loaded rod during an oscillation in which the greatest displacement of the ball from the equilibrium position is 0·15 ft.? What single mass attached directly to the end of the spring, with the rod removed, would give the same period of oscillation? Neglect the effect of the mass of the spring.

(Northern Univ. H.S.C. Schol. level.)

CHAPTER 5

GRAVITATION

5.1 Introduction

In Chapter 2, various units such as force, momentum, work, power and energy were developed out of the basic units mass, length and time. It is interesting to look for a while at the history of the development of these units, and at the experimental work on which Newton's laws were based.

Up to the time of Galileo (1564–1642) bodies were thought to be either heavy or light, they were said to possess 'gravity' or 'levity', and were supposed to fall or rise with velocities proportional to this gravity or levity. Galileo, with his famous experiments conducted from the Leaning Tower of Pisa, demonstrated that this was not true, but that all bodies fall the same distance from rest in the same time, whatever their 'gravity'.

It is doubtful whether Galileo carried out these experiments in the manner usually described, or even whether he was the first person to reach the correct conclusion. Two Dutchmen, Stevin and de Groot, working at Delft in about 1590, dropped two spheres, one made of lead and the other of wax, from a tower and noted that they reached the ground very nearly at the same time, the small difference they attributed quite correctly to air resistance. Galileo had performed similar experiments in his youth, but seems to have misinterpreted them, for in his early writings he records that 'if a lump of lead and a lump of wood are dropped together from a very high tower, the lead will very soon leave the wood behind'. In 1591, however, Galileo repeated these experiments and this time records that 'a cannon ball falls no faster than a musket ball'.

Galileo then set out to find how the speed of the falling body varied during its fall. His first suggestion, that the speed of the body was proportional to the distance through which it had fallen, proved wrong, and he then suggested that the speed was proportional to the time for which it had been falling. Various experiments proved that this suggestion was correct and from it Galileo deduced that a body falls with constant acceleration. We should deduce this fact from the experiment in the following manner.

Experiment shows that speed is proportional to time of fall or $v \propto t$, thus $v = kt$ where k is the constant of proportionality.

Differentiating with respect to time gives $\dfrac{dv}{dt} = k$, but dv/dt is the acceleration a, thus $a = k$, or the acceleration of a falling body has a

constant value. Further, since all bodies fall equally fast, they must all fall with the *same* acceleration.

This acceleration we now know to be due to the gravitational pull exerted by the Earth; it is called 'the acceleration due to gravity' and is represented by the symbol g. The value of g is very nearly a constant over the surface of the Earth, and is usually taken as 981 cm.sec^{-2} or 32·2 ft.sec^{-2}, but varies as we leave the surface of the Earth.

Galileo, in his experiments to show the proportionality of speed to time of fall, found the speeds in practice too large to measure. He tried to reduce these speeds by experimenting with inclined planes, and discovered that the speed of a small ball rolling down a smooth plane with very little friction was the same as that of a ball which had fallen through the same vertical distance. This enabled him to experiment over the equivalent of very small vertical distances and thus at slow speeds which he could measure accurately.

One of the oldest scientific ideas which we possess is that an effort is needed to move a body—a push is needed to start a boulder rolling, a pull is needed to move a cart, etc.—to this effort the name 'force' is given and our experience tells us that a force is needed to produce motion. Conversely, since a body falls, i.e. acquires downward motion, then some force must be pulling it downwards; in fact if the body is held in the hand, this force can be felt pulling the body down—we call it the 'weight' of the body. Galileo, whilst realising that such a force must exist, had no idea of its cause, but by further experiments he was able to discover a number of its effects. For example, he discovered that a ball, if allowed to roll down one incline, would climb to the same *vertical* height up another ramp whatever its inclination. He concluded from this that just as force (i.e. weight) could produce motion, so it could also destroy motion. He ascribed to the body a quality called *Inertia*, which normally tried to keep it at rest, but which could be overcome by a force, so putting the body into motion. Once the body had picked up speed under the action of a force, then its inertia would keep it moving and the body could be brought to rest only by the action of an opposing force. Further, he noticed that if the ball ran from a ramp on to a horizontal plane, then the ball moved along this plane in a straight line with undiminished velocity. In this case the motion of the ball is presumably not being influenced by its weight (which produced motion only in the vertical direction) and thus the ball continues to move whilst under the action of no force at all. This was completely contrary to the ideas of the time, for a constant speed was thought to need a constant force to maintain it (probably due to the excessive friction present in most appliances of those times).

Further, Galileo concluded from this experiment that the natural path of a body in motion is a straight line and that a force is needed to make a body deviate from the straight line. This again was quite a

revolutionary idea. The paths of planets around the Sun were known to be nearly circular; since they were heavenly and rather mystic bodies, their circular orbits were regarded as 'natural' paths and all bodies moving freely were expected to follow similar 'natural' paths, i.e. move in circles. Galileo's theory required that these planets should move naturally in a straight line and he explained their circular orbits by assuming that a force acts continually on them, so bending the orbits into circles. He was unfortunately unable to explain how this necessary force was produced, and so his theories received little credence.

Galileo did not express the results of his experiments and his conclusions drawn from them in the fashion to which we are accustomed today; this was left to Newton (1642–1727), who summarised the results of Galileo's work in the first two laws of motion and then added a third law from his own experience. Newton also propounded the theory of gravitation, discussed later in this chapter, which accounts for the weight of a body and also explains the origin of the force needed by Galileo to bend the orbits of the planets into circles.

5.2 The Laws of Motion

Newton stated his laws of motion as follows:

I. Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.

II. The change of motion is proportional to the motive force impressed and takes place in the direction of the straight line in which that force acts.

III. To every action there is always opposed an equal reaction; or the mutual actions of two bodies upon each other are always equal and oppositely directed.

Newton supported these laws with a number of basic definitions which add a certain amount of clarity.

Definition I

The quantity of matter arises from its volume and density conjointly, thus if we wish to double the density of air and at the same time double its volume, we shall need four times the quantity of air. . . . It is this quantity that is here called 'body' or 'mass'.

Newton thus thought of *mass* as being the quantity of matter in a body, and defined mass as density times volume. (We shall see later that this definition is nowadays regarded as unsatisfactory.)

Definition II

The quantity of motion arises from the velocity and quantity of matter conjointly.

In conjunction with Definition I, it becomes evident that by 'motion', Newton meant the product (velocity $\times$ mass); this quantity we now call momentum.

Further, from the rest of Newton's work, it is clear that although he wrote 'change of motion', the idea that he had in mind was '*rate of change* of motion'. (Newton at this time was also developing the differential calculus and so was well acquainted with 'rates of change'.)

Newton's second law then reads:

'The *rate of change of momentum* is proportional to the motive force impressed.'

5.3 Criticism of Newton's Laws

Newton's laws served as a basis for dynamics without serious criticism for two hundred years; in 1883, however, Mach pointed out that they had certain inherent defects. Firstly, Newton defines mass as volume times density, whereas we can only define density as mass per unit volume, and hence we have no real definition either of mass or density. Further, although the first law gives a description of force (i.e. as that which produces a change of motion in a body), the second law, which tells us how to measure a force (by the rate of change of momentum it produces), leads to the equation:

$$f = ma,$$

and since we have no definition of mass, then we have no definition of force either.

If mass is to be used as a fundamental unit, the essential need is not so much to produce a clear-cut definition of mass as to make a sample unit mass which can be preserved for all time, and to define a method by which all other masses can be compared with the standard.

Newton noted that the masses of two bodies could be compared by means of their weights and this is the universally accepted laboratory method of comparing masses; it is open, however, to the theoretical objection that the weight of a body is really only the gravitational pull exerted on it by the Earth, and we can thus use this method of comparing masses only if the two masses are in a place where the Earth's (or any other) gravitational pull on each is equally effective. For example, even if we could construct a suitable balance, weighing would not serve to compare the masses of two bodies, one situated on the Earth and the other on the Moon, since they would be subjected to quite different gravitational pulls. Moreover, weighing fails completely to tell us the mass of the Earth itself; the Earth has no weight in the accepted sense since it exerts no gravitational pull *on itself*, nevertheless it possesses plenty of mass, i.e. 'quantity of matter', about 6×10^{24} kilogrammes in fact! It must be remembered that these objections are purely of a theoretical nature, and in no way invalidate the weighing process as usually performed.

We have already seen that mass can be considered as the inertia of a body, or the property of a body which makes it resist a force trying to move it. Using this idea of mass, Mach proposed the following argument.

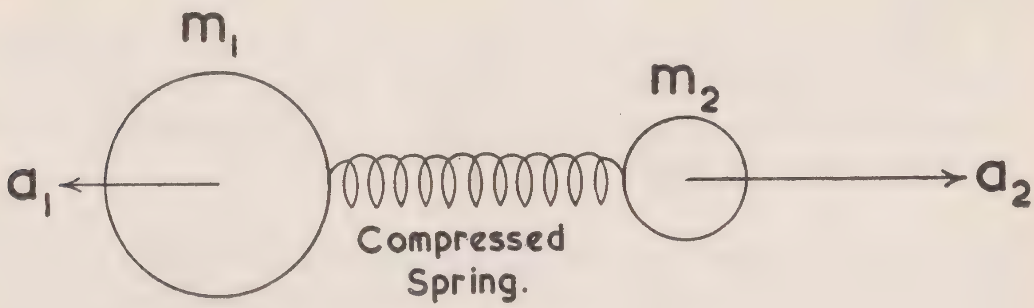


Fig. 5.1

If two bodies act on each other in any way, whether by means of gravitational or of magnetic attraction, or if a piece of elastic stretched between them pulls them together, then it is observed as an experimental fact that the bodies move with accelerations which are in a constant ratio, the larger body always having the smaller acceleration. The size of the ratio depends only on the bodies and is quite independent of whatever is used to accelerate them. The property of the body which most naturally accounts for this fact is the inertia of the body, and this we have already linked with mass. We can then define the ratio of the masses of the two bodies as the inverse ratio of the accelerations which they produce in each other; from Fig. 5.1:

$$\frac{m_1}{m_2}=\frac{a_2}{a_1} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Whether or not this process defines the same sort of mass as Newton had in mind is really beside the point. Mach was concerned with setting the science of dynamics on a firm logical footing, and as such was really starting afresh. He could thus define mass in any way that he found convenient; of course, if his definition should lead to others already in use and be consistent with them, so much the better. Mach claimed that the idea he had taken as the basis was the only fundamental principle emerging from the work of Galileo and Newton, and the various laws of motion were deductions from this principle.

The development of the theories of dynamics can now follow from Mach's principle on these lines.

1. Relative masses can be deduced from accelerations using the expression

$$\frac{m_1}{m_2}=\frac{a_2}{a_1}.$$

2. Experiment shows that the relative masses of two bodies remain constant to a high degree of accuracy.

3. We can then accept mass as a fundamental unit in addition to length and time, and can make and preserve a standard unit mass.

4. Acceleration experiments then enable us to make multiples of the standard unit mass and to measure masses in terms of the standard units.

5. Next it is recognised that some agency is needed to produce motion in a massive body; this agency is named 'force' and it is to be measured by the product of the mass of a body and the acceleration that it receives, thus:

$$f = ma \quad . \quad . \quad . \quad . \quad (2)$$

This method of measuring a force does not appear to have a logical backing, but rests on experimental evidence, for experiments with a constant force (for example, a spring always compressed to the same extent), used to accelerate bodies of various masses, would quickly show that the acceleration is inversely proportional to the mass or

$$a \propto \frac{1}{m}$$

$$= \frac{k}{m},$$

where k is the constant of proportionality.

Thus $$ma = k,$$

and it is very reasonable to take this constant k as a measure of the constant force used to produce the acceleration, leading straight away to $f = ma$ as in Equation (2).

This expression for force is also in accord with well-established ideas, for:

$$a = \frac{dv}{dt},$$

$$\text{therefore } f = m\frac{dv}{dt}$$

$$= \frac{d}{dt}(mv) \quad . \quad . \quad . \quad (3)$$

The product (mv) we can name 'momentum' and thus force is measured by the rate of change of momentum as in Newton's second law, but since force is already defined, this expression can now be used to define momentum.

6. If two bodies of masses m_1, m_2 act on each other so as to produce accelerations a_1, a_2, then the force acting on the first body is given by:

$$f_1 = m_1 a_1$$

and on the second by:

$$f_2 = m_2 a_2,$$

but, from Mach's principle:

$$\frac{m_1}{m_2} = \frac{a_2}{a_1}$$

$$\text{or} \quad m_1 a_1 = m_2 a_2,$$

$$\text{therefore} \quad f_1 = f_2.$$

Thus equal forces are exerted on the two bodies by each other as described in Newton's third law.

7. If two bodies of masses m_1, m_2 have weights W_1, W_2 and fall with accelerations a_1, a_2, then, remembering that weight is a force,

$$W_1 = m_1 a_1,$$

$$W_2 = m_2 a_2,$$

$$\text{therefore} \quad a_1 = \frac{W_1}{m_1},$$

$$a_2 = \frac{W_2}{m_2}.$$

But experiment shows that all bodies fall with the same acceleration, i.e.

$$a_1 = a_2,$$

$$\text{therefore} \quad \frac{W_1}{m_1} = \frac{W_2}{m_2}$$

$$\text{or} \quad \frac{m_1}{m_2} = \frac{W_1}{W_2} \qquad (4)$$

and thus the weight of a body is proportional to its mass.

To summarise, then, Mach objects to Newton's laws because Newton describes mass (as quantity of matter) but advances no method of *measuring* it; he then proceeds to describe momentum in terms of mass and velocity, since mass cannot be measured neither can momentum. Finally Newton describes force and says it is to be measured as 'rate of change of momentum', but again, since momentum cannot be measured, neither can force.

Mach avoids this trouble by describing an experiment by which the mass of a body may be measured in terms of a standard mass (incidentally, notice that this measurement involves an acceleration only, which can be measured in terms of length and time—the other two basic units). He then describes force (the agency that causes a mass to accelerate) and gives a method of measuring force, in terms of mass and acceleration, which can be demonstrated by experiment to be consistent. Next, the relation between force and momentum is derived and used as a definition of momentum and finally the proportionality of mass and weight is demonstrated.

5.4 Newton's Method for Demonstrating Proportionality of Mass and Weight

Newton used a pendulum to demonstrate the proportionality of mass and weight; if we assume no relation to hold between mass and weight, the theory of the simple pendulum as given on page 101 must be modified as follows.

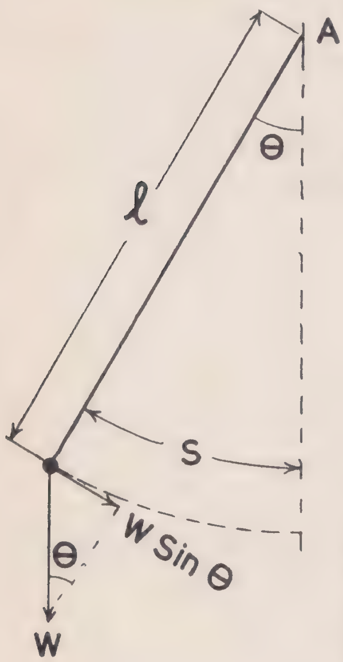


Fig. 5.2

The forces acting on the bob (Fig. 5.2) are its weight W and the tension in the string, but only the component of the weight perpendicular to the string has any restoring effect, thus:

$$\text{Restoring force} = W \sin \theta.$$

If θ is small, this can be written as:

$$\text{Restoring force} \simeq W\theta$$

$$= \frac{Ws}{l}.$$

Thus if W is a constant, the restoring force is proportional to the arc displacement s and thus the motion is simple harmonic. The restoring force at unit displacement is then equal to W/l and if the bob is of mass m, the period is given by:

$$\tau = 2\pi \sqrt{\frac{m}{W/l}}.$$

$$\text{Thus } \frac{W}{m} = \frac{4\pi^2 l}{\tau^2} \qquad . \quad . \quad . \quad . \quad (5)$$

Now if weight is proportional to mass, W/m must be constant, thus the factor $\left[\frac{4\pi^2 l}{\tau^2}\right]$ for a pendulum should be constant *whatever the mass of the bob.* Newton made a pendulum having a hollow box as a bob, the box he filled with different magnitudes and sorts of masses, i.e. lead, sand, grains of corn, and found that the factor $(4\pi^2 l/\tau^2)$ did in fact remain constant. Bessel repeated this experiment at the end of the last century and confirmed Newton's result to a high degree of accuracy.

The reason for using a hollow box as the bob was to ensure that the effects of air resistance, which depend at a given velocity only on the shape and size of the bob, would be the same in each experiment, and so would not invalidate the results.

5.5 Gravitation

Galileo, in order to explain the motion of the planets, postulated the existence of a force which continually deflected each planet from its natural path (i.e. a straight line) into a circular orbit. It seems

obvious to us to ascribe this force to the gravitational attraction between the Sun and the planets. Galileo and his contemporaries had worked on many aspects of gravity (e.g. the motion of a falling body), but their concepts were very different from those we hold today.

Copernicus, for example, thought of gravity as some mystic property of the *centre* of the Earth. He described it merely as the tendency of a body to get back to the centre of the Earth if removed from it. Although he considered all heavenly bodies to have gravity, he thought of each as having its own sort of gravity. Thus a body removed from the Sun would attempt to get back to the Sun and would have nothing to do with the Earth and so on. The essential idea which was missing from this concept of gravitation was that one body could exert a *force* on the other.

A large number of experiments was being performed at this time, however, and many scientists were striving to gain an understanding of the motion of the heavenly bodies. Gradually the evidence was produced which enabled Newton to enunciate a law of gravitation. The first real advance came from Gilbert (1540–1603). He was experimenting with magnetism and made a small magnetised sphere which, he realised, behaved in much the same way as the Earth. Although he was thereby led wrongly to explain the Earth's gravity as magnetism, he did stumble on two important facts. Firstly, gravity is a reciprocal effect; that is, if the Earth exerts an attractive force on a body, then the body exerts the same attraction on the Earth. Secondly, gravity is not a property of a point in the body but of the whole matter in the body; there is no especial property concerning the *centre* of the Earth, but the gravitational effects of all the particles making up the Earth *appear to act at the centre* since the Earth is a symmetrical body.

Many experiments were devised to test these theories. Francis Bacon, for example, pointed out that if the second part of Gilbert's theory were true, then the attraction due to gravity should decrease in a mine since there would be less matter beneath to cause a downward attraction. Hooke actually carried out this experiment but with inconclusive results. So far these theories are restricted to effects occurring on the Earth, but Robertval extended the idea of gravitation to the whole universe and stated a theory of universal gravitation—that every particle of matter exerted an attractive force on all other particles wherever they might be. Unfortunately he could formulate no mathematical laws to describe the effects of gravitation.

The idea of universal gravitation had many opponents, and one of the objections they proposed was this—if the gravity of the Earth attracts the Moon, what prevents the Moon from falling into the Earth? The answer to this was provided in 1665 by Borelli, who recognised that a centripetal force is needed to maintain motion in a circle; gravity provides this centripetal force. The Moon *does* fall towards the Earth, but

only enough to keep it in its circular orbit. This is illustrated in Fig. 5.3. Borelli, whilst describing centripetal force, could not deal with it mathematically; this was done by Huygens and, independently, by

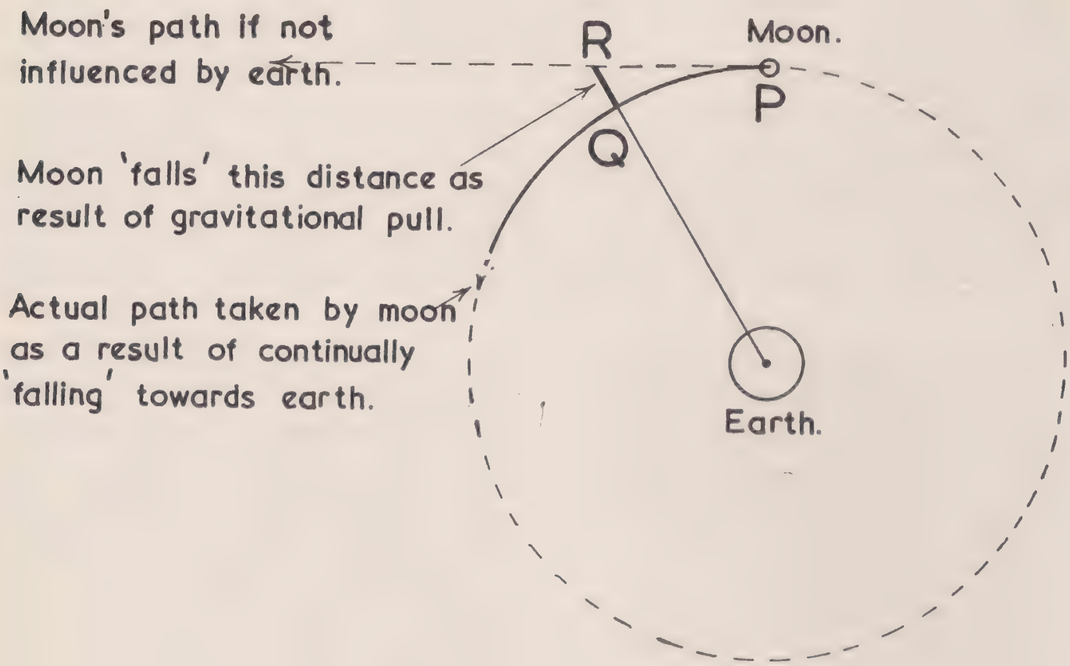


Fig. 5.3

Newton, who once again collected up the various pieces of information available at the time and from them stated his law of universal gravitation.

First of all Newton deduced that the gravitational force between two masses was inversely proportional to the square of the distance between them (this fact seems also to have been deduced by Hooke, Hadley, Huygens and Christopher Wren working together); the evidence on which he had to work consisted of his own discoveries concerning centripetal force and Kepler's law of Planetary Motion. These were laws which Kepler had found by trial and error to agree with Tycho Brahé's observations of planetary motion, and are usually stated as follows.

5.6 Kepler's Laws

(1) Each planet describes an elliptical orbit with the Sun at one of the foci.

(2) The line drawn from the sun to the planet sweeps out equal areas in equal times, i.e. if *ABCDE* (Fig. 5.4) are the positions of the planet in its orbit on successive days, then the sectors *ASB*, *BSC*, *CSD*, etc., all have the same area.

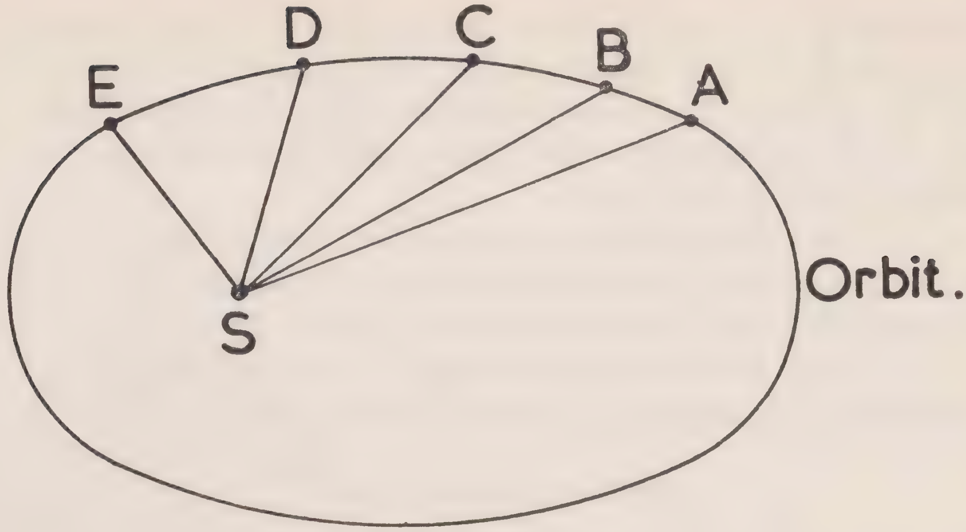


Fig. 5.4

(3) The square of the time taken for a planet to go once around its orbit is proportional to the cube of the major axis of the orbit.

5.7 Derivation of Inverse Square Law of Gravitation

First of all, assume that the planetary orbits are circles and not ellipses. This is not a bad approximation, since, apart from Pluto, the outermost of the planets so far discovered, the ratio of the major to the minor axis does not exceed 1·02 for any planetary orbit.

If v is the velocity of the planet in its orbit, then the time to go round once is equal to $2\pi r/v$, if r is the radius of the orbit; therefore from Kepler's last law $4\pi^2r^2/v^2$ is proportional to r^3 or:

$$v^2 \propto \frac{1}{r} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

But since the planet is moving in a circle, it has a radial acceleration of v^2/r (see page 85) and combining this with Equation (6) gives:

$$\text{Radial Acceleration} \propto 1/r^2 \quad . \quad . \quad . \quad (7)$$

But we know that force = mass × acceleration, thus, for a constant mass, force is proportional to acceleration and in this case, Equation (7) can be written as:

$$\text{Radial force} \propto 1/r^2.$$

But the radial force which bends the orbit of a planet into a circle is the gravitational force between the planet and the Earth, hence we get finally:

$$\text{Gravitational force} \propto \frac{1}{r^2} \quad . \quad . \quad . \quad (8)$$

5.8 Newton's Law of Universal Gravitation

As the next step in the formulation of the general law of gravitation, Newton turned his mind to the effect of the masses of the bodies on the gravitational force between them.

The weight of a body is really the gravitational force exerted on it by the Earth, but we also know that the weight of a body is proportional to its mass (Newton assured himself of the truth of this statement by the experiments with a pendulum, described earlier in this chapter), thus we know that:

$$\text{Gravitational force} \propto \text{mass of attracted body} \quad . \quad (9)$$

and if we accept that gravitation is a reciprocal process, i.e. both bodies of a gravitating pair attract each other equally, then:

$$\text{Gravitational force} \propto \text{mass of attracting body} \quad . \quad (10)$$

Combining Equations (8), (9) and (10) then gives:

$$\text{Gravitational force} \propto \frac{m_1 m_2}{r^2} \quad . \quad . \quad . \quad (11)$$

where m_1, m_2 are the masses of the two attracting bodies. Thus we may write:

$$\text{Gravitational force} = \frac{G m_1 m_2}{r^2} \quad . \quad . \quad . \quad (12)$$

where G is the constant of proportionality and is known as the *Universal Gravitational Constant* (it is important that this constant should not be confused with g, the acceleration due to gravity).

The deduction of this law is obviously mainly intuitive, but Newton tested its truth by many experiments and calculations. Firstly, starting only from his law of gravitation, he was able to derive mathematically all of Kepler's laws (which are experimental laws deduced from astronomical observations) and was able to predict that the orbits should be ellipses (although in formulating the gravitational law we assumed the orbits to be circles).

This test shows only the inverse square part of the law to be true, but a proof of the fashion in which mass enters into the law is provided by the proportionality of mass and weight as explained above.

Finally, Newton tested the universality of the law (i.e. is G the same for all gravitating systems?) by comparing the gravitational effect of the Earth on the Moon with the effect on a body falling near the surface of the Earth. Referring again to Fig. 5.3, if the force of gravity could be 'switched off' when the Moon reaches the point P in its orbit, then in one second the Moon would move along the straight line to R instead of being bent around to Q; in fact, the effect of gravity is to make the Moon fall the distance RQ towards the Earth in one second. Newton knew that the radius of the orbit of the Moon was about 240,000 miles and that it took $27\frac{1}{3}$ days to complete one circuit of its orbit, thus he was able to calculate RQ as 0·0044 feet. Now the radius of the Earth is about 4000 miles, consequently, if the inverse square law is true, then the gravitational force exerted on a body falling near the surface of the Earth should be $\left(\frac{240{,}000}{4{,}000}\right)^2$ times greater than the gravitational force

on the Moon, i.e. 3600 times greater. In one second, therefore, the body near the surface of the Earth should fall from rest through a distance 3600 times greater than does the Moon if they are both subject to the same sort of gravitation, i.e. if the gravitational constant is the same for both.

Now $3600 \times 0{\cdot}0044 \simeq 16$, and experiment shows that on the Earth a body falls 16·1 ft from rest in one second—Newton records that his theory and experiment agreed 'pretty nearly'!

Later experiments have shown that G is a universal constant to a high order of accuracy, unlike g, the acceleration due to gravity, which varies from place to place.

Newton reached this stage in 1666, but did not publish his work immediately, as he appears to have had some doubts concerning the validity of the approximation he had introduced in assuming the whole of the mass of the Earth to be concentrated at its centre. In 1684, however, Newton was able to *prove* that a sphere could be regarded in this way and that his previous work had in fact not involved an approximation at all; he therefore felt justified in publishing his theory of gravitation with his famous *Principles* in 1687.

Example 1. (*a*) *At what velocity must a body be projected horizontally on the surface of the Earth in order that it shall perform a circular orbit just above the Earth's surface? (Gravitational constant* $= 6{\cdot}66 \times 10^{-8}$ *c.g.s. units, Mass of Earth* $= 5{\cdot}98 \times 10^{27}$ *gm, radius of Earth* $=$ 6370 *km). Neglect air resistance.*

(*b*) *At what speed must a body travel to perform for ever a circular orbit at a distance of* 1650 *kilometres from the Earth's surface (i.e. to become an artificial satellite)?*

(*a*) Let the body be of mass m, the Earth of mass M and radius R: then the gravitational force on the body is given by:

$$F = \frac{GmM}{R^2}$$

where G is the gravitational constant.

The body will perform a circular orbit of radius R with velocity v if subjected to a centripetal force mv^2/R: this is provided by the gravitational force, hence:

$$\frac{mv^2}{R} = \frac{GmM}{R^2},$$

$$\text{or } v = \sqrt{\frac{GM}{R}}$$

$$= \sqrt{\frac{6{\cdot}66 \times 10^{-8} \times 5{\cdot}98 \times 10^{27}}{6{\cdot}37 \times 10^{8}}} \text{ cm.sec}^{-1}$$

$$= 7{\cdot}9 \text{ km.sec}^{-1}$$

(*b*) As in the previous example,

$$v = \sqrt{\frac{GM}{R}},$$

where R is now $6370 + 1650$

$$= 8020 \text{ km.}$$

$$\text{Thus } v = \sqrt{\frac{6{\cdot}66 \times 10^{-8} \times 5{\cdot}98 \times 10^{27}}{8{\cdot}020 \times 10^{8}}}$$
$$= 7{\cdot}05 \text{ km.sec}^{-1}.$$

5.9 Gravitational Force due to Spherical Bodies

(a) Spherical Shell

Consider first of all a thin spherical *shell* of radius r and thickness δt, let us calculate the gravitational attraction that it exerts on a mass m at a point P, distance d from the centre of the shell (Fig. 5.5).

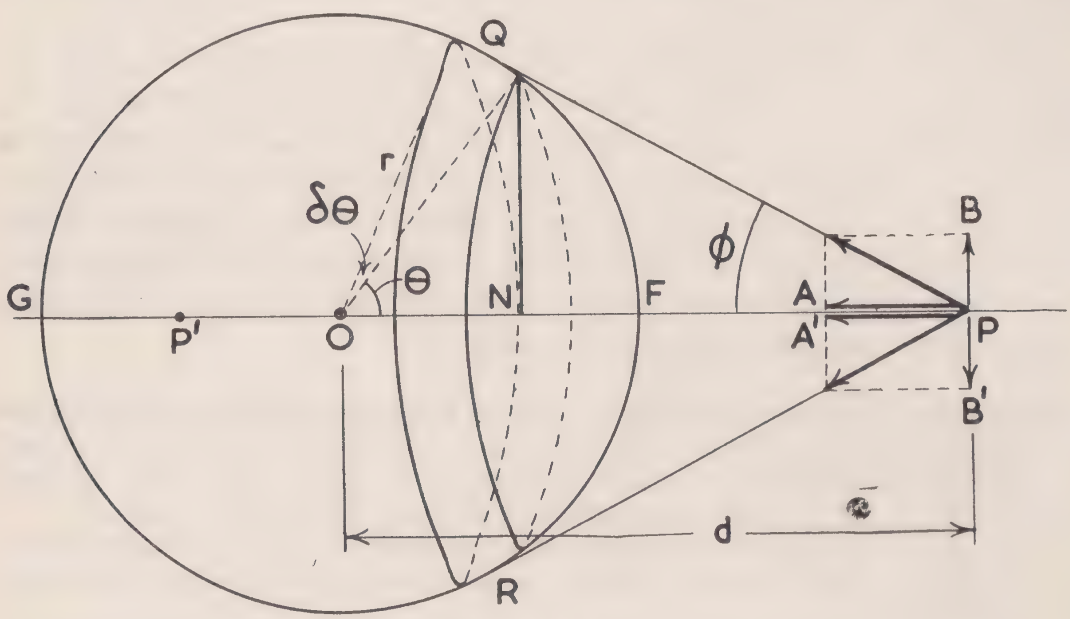


Fig. 5.5

Divide the shell into a number of rings such as QR, the plane of the rings being perpendicular to OP and each subtending an angle $\delta\theta$ at the centre of the sphere, the width of each ring is thus $r\delta\theta$.

Again divide this ring up around its circumference into segments of length δs, then the area of one segment is $r\delta\theta \, . \, \delta s$ and its volume is $r\delta\theta\delta s\delta t$; thus if the shell is of density ϱ, the mass of one segment is $\varrho r\delta\theta\delta s\delta t$ and hence the attraction, δf, exerted by one segment on the mass m at P is given by:

$$\delta f = \frac{Gmr\varrho\delta\theta\delta s\delta t}{x^2} \quad . \quad . \quad . \quad . \quad (13)$$

where x is the distance from the segment to P.

If the segment is situated at Q it will exert the force calculated above along the direction PQ. This force can be resolved into the forces PA along PO and PB perpendicular to PO. A similar segment diametrically opposite Q at R, will exert the same force but in the direction PR and this can be resolved into PA' and PB'. Now PB and PB' will cancel out as they are in opposite directions, and this will happen for every pair

of segments all around the ring. The vector PA, however, will be reinforced by PA', and this will happen for every pair of segments; hence the only force exerted by the ring on the mass m is along PO (as would be expected from symmetry) and can be found by adding up the contributions such as PA for every segment in the ring.

$$\text{Now } PA = \frac{Gmr\varrho\delta\theta\delta s\delta t}{x^2}\cos\varphi,$$

thus total force f on the mass m due to ring is given by:

$$f = \Sigma\frac{Gmr\varrho\delta\theta\delta s\delta t\cos\varphi}{x^2},$$

where the summation has to be taken all round the ring.

Every term in this expression except δs is a constant *for any particular ring*, thus:

$$f = \frac{Gmr\varrho\delta\theta\delta t\cos\varphi}{x^2}\Sigma\delta s.$$

But $\Sigma\delta s$ is merely the circumference of the ring, i.e. $2\pi . QN$, which is equal to $2\pi r\sin\theta$, therefore the force due to the ring on the mass m is given by:

$$f = \frac{Gmr\varrho\delta\theta\delta t\cos\varphi\ .\ 2\pi r\sin\theta}{x^2}$$

$$= \frac{2\pi r^2\,Gm\varrho\delta t\cos\varphi\sin\theta\,\delta\theta}{x^2}.$$

$$\text{But } \cos\varphi = \frac{NP}{QP}$$

$$= \frac{d - r\cos\theta,}{x}$$

$$\text{thus } f = \frac{2\pi r^2\,Gm\varrho\delta t\sin\theta\,\delta\theta}{x^2}\times\frac{(d - r\cos\theta)}{x} \quad . \quad (14)$$

To find the force due to the whole shell, the effect of all the rings must be summed; this is best done by integration. Unfortunately, however, Equation (14) contains two variables, x and θ, one of which must be eliminated before the integration can be carried out. Experience shows that it is best to work in terms of x, θ being eliminated as follows.

$$\text{From the triangle } OQP,\ x^2 = r^2 + d^2 - 2rd\cos\theta \quad . \quad (15)$$

Rearranging this gives:

$$d^2 - 2rd\cos\theta = x^2 - r^2,$$

and adding d^2 to each side and dividing by $2d$ gives:

$$(d - r\cos\theta) = \frac{x^2 - r^2 + d^2}{2d} \quad . \quad . \quad (16)$$

Differentiating Equation (15) with respect to θ gives:

$$2x(dx) = 2rd \sin \theta d\theta$$

$$\text{or } \sin \theta d\theta = x(dx)/rd \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Substituting from Equations (16) and (17) in Equation (14) gives:

$$f = \frac{2\pi r^2 \, Gm \, \varrho \delta t}{x^2} \times \frac{x(dx)}{rd} \times \frac{x^2 - r^2 + d^2}{2xd}$$

$$= \frac{\pi r G m \varrho \delta t}{d^2} \times \left[\frac{x^2 - r^2 + d^2}{x^2}\right] dx.$$

Thus the total attractive force, F, due to the whole shell, is given by:

$$F = \frac{\pi r G m \varrho \delta t}{d^2} \int \left[1 + \frac{d^2 - r^2}{x^2}\right] dx \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where the integral has to be taken between suitable limits to include the whole of the sphere; these limits are the points F and G in the diagram, for which x is equal to $d - r$ and $d + r$ respectively, hence:

$$F = \frac{\pi r G m \varrho \delta t}{d^2} \int_{d-r}^{d+r} \left[1 + \frac{d^2 - r^2}{x^2}\right] dx$$

$$= \frac{\pi r G m \varrho \delta t}{d^2} \left[x - \frac{d^2 - r^2}{x}\right]_{d-r}^{d+r}$$

$$= \frac{\pi r G m \varrho \delta t}{d^2} \left[(d + r) - \frac{d^2 - r^2}{d + r} - (d - r) + \frac{d^2 - r^2}{d - r}\right]$$

$$= \frac{\pi r G m \varrho \delta t}{d^2} \left[(d + r) - (d - r) - (d - r) + (d + r)\right]$$

$$= \frac{4\pi r^2 G m \varrho \delta t}{d^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The area of the surface of the shell is $4\pi r^2$, thus its volume is $4\pi r^2 \, \delta t$ and its mass is $4\pi r^2 \varrho \delta t$; writing this as M, Equation (19) becomes:

$$F = \frac{GmM}{d^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

This is the formula for the gravitational force between two point masses M and m at distance d apart, hence the gravitational attraction produced by a spherical shell is the same as if the whole of its mass were concentrated at its centre.

(b) A Solid Sphere

A solid sphere is made up of a number of concentric shells, each of which behaves as though its mass were concentrated at the centre; hence the whole sphere behaves in the same fashion, which is the result that Newton required. The gravitational force is then given by Equation (20) where M is the total mass of the sphere. It must be emphasised

that this result is true only for a sphere—in general it is not permissible to calculate the gravitational effect of a body by assuming that the whole of the mass is concentrated at the centre of gravity.

(c) Gravitational Force inside a Spherical Shell

Another interesting result can be obtained from the same piece of mathematics. In the section above, the point P was outside the shell; it is possible, however, in the same fashion to calculate the gravitational force on a mass m placed at a point such as P' (Fig. 5.5) *inside* the shell. The working will be identical with that given above as far as Equation (18). This again has to be integrated between the points represented by F and G on the diagram. Remember that x is the distance measured from P' and that $OP' = d$, then at F we have $x = r + d$ and at G, $x = r - d$. Thus:

$$F = \frac{\pi r G m \varrho \delta t}{d^2} \int_{r+d}^{r-d} \left[1 + \frac{d^2 - r^2}{x^2}\right] dx$$

$$= \frac{\pi r G m \varrho \delta t}{d^2} \left[x - \frac{d^2 - r^2}{x}\right]_{r+d}^{r-d}$$

$$= \frac{\pi r G m \varrho \delta t}{d^2} \left[(r-d) - \left(\frac{d^2 - r^2}{r-d}\right) - (r+d) + \left(\frac{d^2 - r^2}{r+d}\right)\right]$$

$$= \frac{\pi r G m \varrho \delta t}{d^2} \left[(r-d) + (r+d) - (r+d) - (r-d)\right]$$

$$= 0,$$

i.e. the material of a spherical shell exerts no gravitational attraction on a mass placed anywhere within the shell.

(d) Gravitational Attraction within the Material of a Solid Sphere

The gravitational attraction exerted on a mass at the point P inside a solid sphere can be found by drawing the concentric spherical surface through P shown dotted in the diagram (Fig. 5.6). P is *inside* all the material of the outer shell and hence suffers no attraction due to it, but is outside all the material of the inner core and is thus attracted by this material acting as a solid sphere.

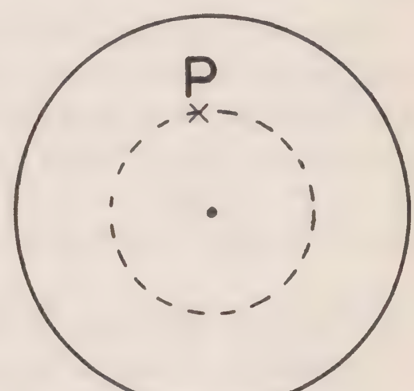


Fig. 5.6

If r is the distance of P from the centre of the sphere and ϱ the density of the material, then the mass of the central core is $\frac{4}{3}\pi r^3 \varrho$; the sphere behaves as though its mass is concentrated at the

centre, hence the gravitational attraction on a mass m at P is given by:

$$\text{Gravitational force} = \frac{Gm \cdot \frac{4}{3}\pi r^3 \varrho}{r^2},$$

$$= \tfrac{4}{3}Gm\pi r\varrho \quad . \quad . \quad . \quad (21)$$

The gravitational force on a body therefore becomes smaller as it penetrates farther into the sphere, i.e. as r decreases. If the sphere is the Earth, this is tantamount to saying that the weight of a body decreases as it penetrates into the Earth due to the decreased gravitational effect.

5.10 Measurement of the Gravitational Constant

Although Newton's work, and innumerable observations made since his time, have all gone to verify the universal gravitational law, and to substantiate the fact that G is a universal constant, none of the experiments so far described enable the value of G to be calculated.

Note that G is a quantity having dimensions, not just a number, for rearranging Newton's law gives:

$$G = \frac{fd^2}{m_1 m_2}$$

$$\text{whence: } [G] = \left[\frac{\text{force} \times \text{dist.}^2}{\text{mass}^2}\right]$$

$$= \left[\frac{MLT^{-2} \times L^2}{M^2}\right]$$

$$= [M^{-1}L^3T^{-2}].$$

The c.g.s. unit of G is thus the $\text{gm}^{-1}.\text{cm}^3.\text{sec}^{-2}$ and although it is a universal constant, its value changes according to the units in which mass length and time are measured.

Note again the difference between G and g, for whereas the dimensions of G are $M^{-1}L^3T^{-2}$, g is an acceleration, and has the dimensions LT^{-2}.

From the equation above it is seen that G is numerically equal to f if m_1, m_2 and d are all equal to unity, i.e. G is numerically equal to the gravitational force between two 1 gm masses when they are 1 cm apart. Experience tells us that this force is rather small (in fact it is about $6{\cdot}6 \times 10^{-8}$ dynes or $6{\cdot}75 \times 10^{-11}$ gm-wt) and we can anticipate some difficulty in measuring it!

(a) Torsion Balance Method of measuring *G*

Cavendish, using a torsion balance, succeeded in measuring the force between two small masses as early as 1798. In 1894 his method was elaborated by C. V. Boys into one of high accuracy.

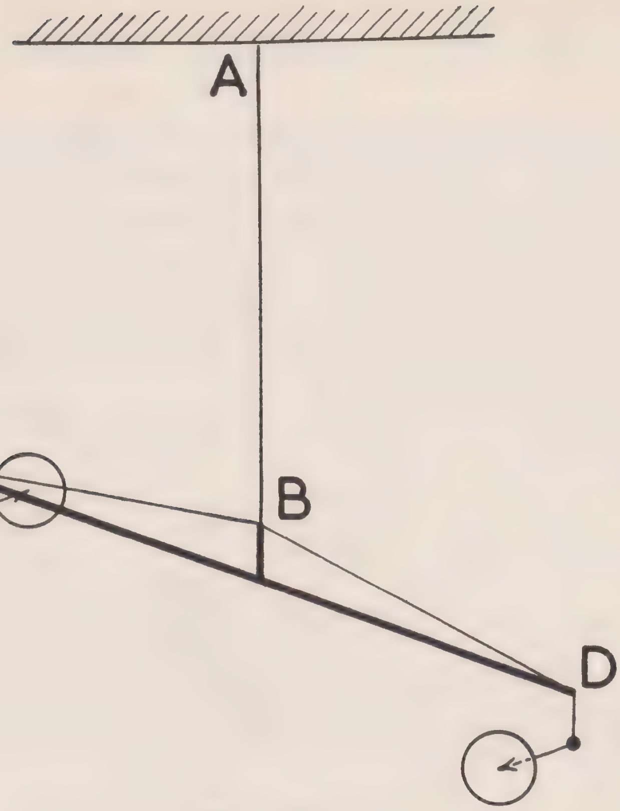


Fig. 5.7

A torsion balance is a piece of apparatus in which a couple is measured by the twist which it produces in a wire. (See Chapter 8.) The balance used by Cavendish consisted of a wire AB (Fig. 5.7) carrying a beam CD, this in turn supported at its ends two equal masses m; two large masses M were moved to alternative positions on either side of the beam as shown in the plan view in Fig. 5.8.

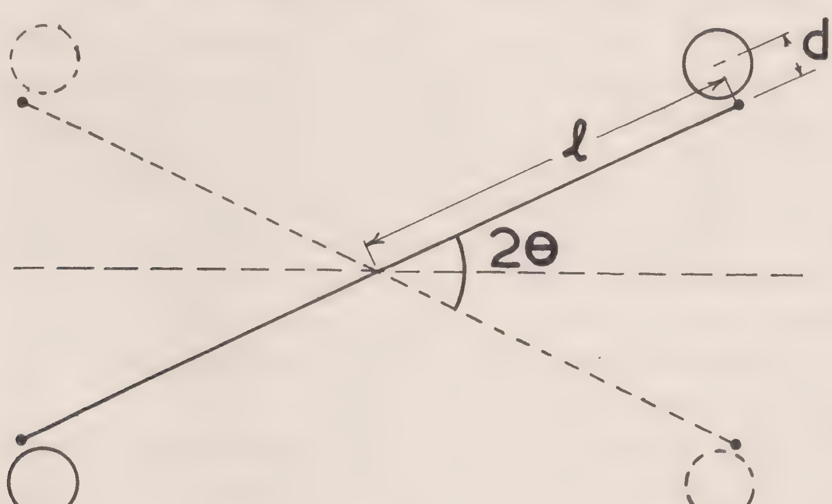


Fig. 5.8

The attractive force between each pair of adjacent masses forms a couple acting on the beam; this rotates, twisting the supporting wire, until the couple produced by the wire is equal to the couple rotating

the beam. By measuring the deflection of the beam from the rest position Cavendish was able to calculate G; the theory of the method is given below.

If, when the beam has come to rest in its deflected position, the distance between the centres of a pair of spheres is d, then the attractive force experienced by either of the small masses is GmM/d^2 and the couple turning the beam is $2GmMl/d^2$. If the torsional rigidity (see Chapter 8) of the suspension is c, i.e. a couple of c dyne.cm is needed to twist it through 1 radian, then the couple to twist it through the angle θ between the rest and deflected positions is $c\theta$.

The beam will come to rest in the position which makes these two couples equal, i.e.:

$$\frac{2GmMl}{d^2} = c\theta,$$

$$\text{hence } G = \frac{c\theta d^2}{2mMl} \quad . \quad . \quad . \quad . \quad (22)$$

The constant c is found from an oscillation experiment; it is shown on page 222 that the time of oscillation of a torsional pendulum is given by:

$$T = 2\pi\sqrt{\frac{I}{c}},$$

where I is the moment of inertia of the suspended system (beam and small masses in this case) and c is the torsional rigidity of the suspension.

$$\text{Thus } c = \frac{4\pi^2 I}{T^2},$$

and this value can be substituted in Equation (22). Usually it is much easier to measure T and calculate I than to measure c directly.

In the apparatus used by Cavendish the suspension wire was of silvered copper 3 ft long, the beam a 6-ft deal lath carrying a 2-in. diameter lead sphere at each end while the deflecting masses were 8-in. diameter lead spheres. The angular deflection was very small, but it was doubled by placing the deflecting masses first on one side of the beam and then on the other and reading the angle between the two rest positions, this also eliminated any error due to asymmetry of the apparatus. The deflection was read by attaching to the ends of the beam verniers moving over fixed scales.

The size of the apparatus made it very susceptible to disturbance by draughts, even though it was enclosed in a room. All adjustments were made by rods through the wall and the readings taken through a telescope. The long time of swing of the beam (7 minutes) meant that it very rarely came permanently to rest. Despite these handicaps, Cavendish obtained the value of $6{\cdot}74 \times 10^{-8}$ c.g.s. units for G, which is only about 2 per cent. higher than the value accepted today.

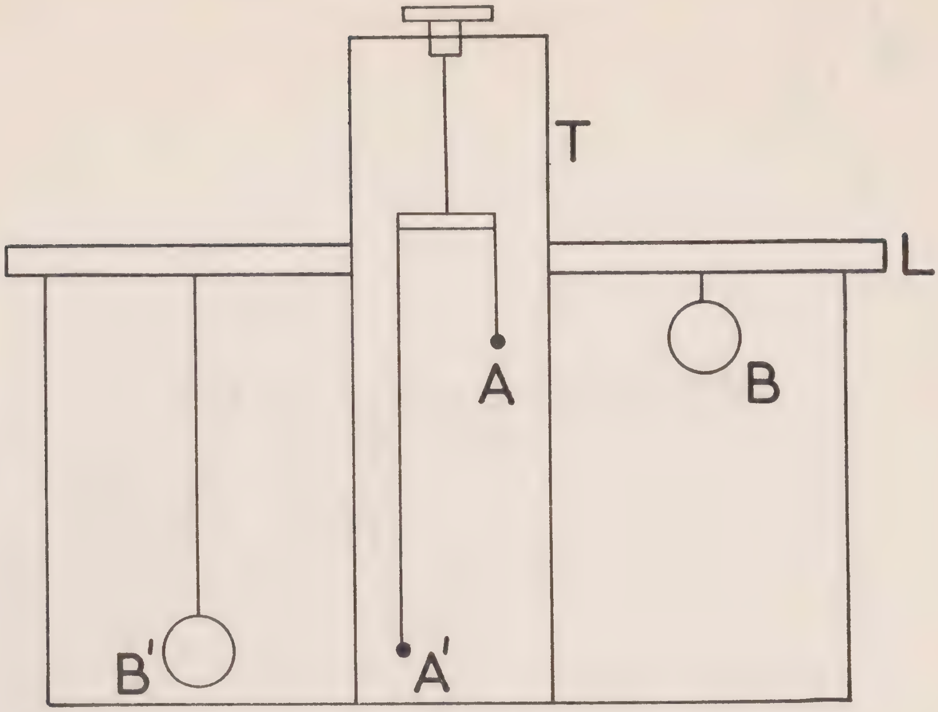


Fig. 5.9

C. V. Boys' apparatus followed the same plan as that of Cavendish, but was much smaller and consequently avoided the trouble due to draughts. The suspension was a drawn quartz fibre 42 cm long. Boys had previously discovered how to draw quartz into fibres as fine as 0·0005 cm diameter; these were capable of supporting considerable weights, but exerted much less resistance to a twist than a wire of the same strength. Various supported masses AA' (Fig. 5.9) were used, the largest being gold spheres 1 cm in diameter, which were hung from the ends of an aluminium beam 2·3 cm long. The beam was silvered and acted as a mirror, its angular deflection being measured by observing through a telescope the image of a scale reflected in the mirror. The complete suspension system was enclosed in a glass tube T to protect it from draughts.

The deflecting masses BB' were lead spheres weighing about $7\frac{1}{2}$ kg each; owing to the closeness of the masses AA', the pairs AB, $A'B'$ were hung at different heights (about 15 in. apart) to reduce the cross-attraction of B' for A and B for A'. The large masses were supported from the lid L which could be rotated; to obtain a reading the lid was rotated until the deflection of the beam was at a maximum; this is rather a different procedure from that adopted by Cavendish and the theory must be modified as follows.

When the deflection of the beam has reached a maximum value φ, let the relative position of the small and large masses be as shown in Fig. 5.10; then if the spheres A, A' are of mass m and B, B' of mass M, the attraction on each small sphere is $\dfrac{GmM}{AB^2}$ directed along AB or $A'B'$,

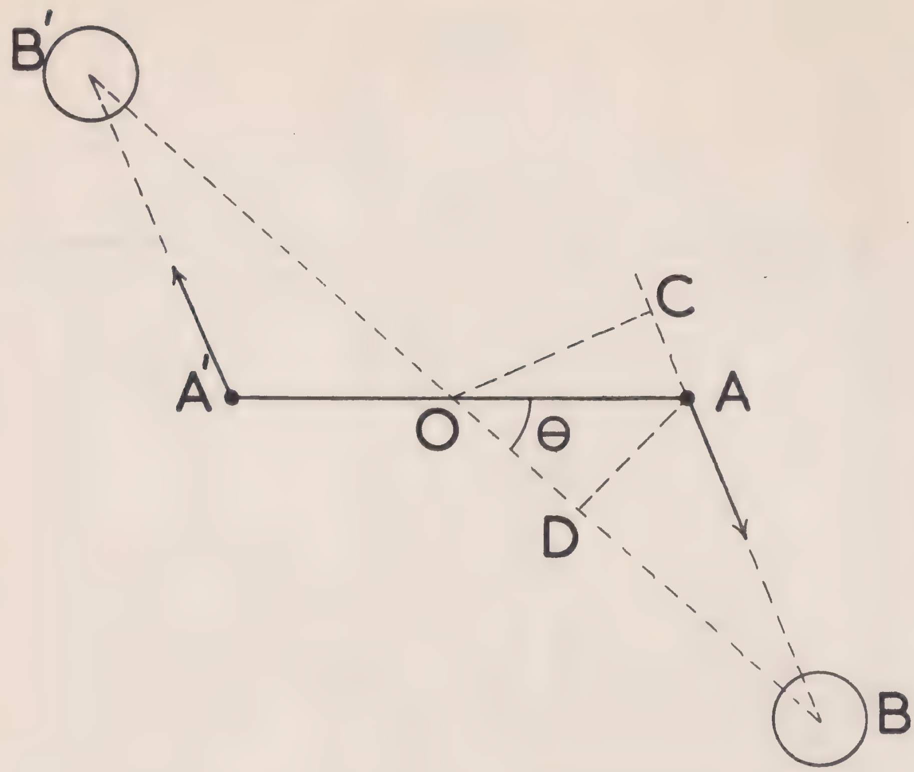


Fig. 5.10

the total couple exerted on the beam is therefore $\left(\frac{2GmM}{AB^2} \times OC\right)$ where OC is drawn perpendicular to AB.

The value of OC can be found as follows. Draw AD perpendicular to OB, then the triangles OCB and ADB are equiangular and therefore similar.

$$\text{Thus } \frac{OC}{OB} = \frac{AD}{AB}$$

If OA, OB are of length l and d respectively, then this ratio can be written as:

$$\frac{OC}{d} = \frac{l \sin \theta}{AB}$$

$$\text{or } OC = \frac{ld \sin \theta}{AB}.$$

Therefore the couple Γ acting on the beam is given by:

$$\Gamma = \frac{2GmMld \sin \theta}{AB^3}$$

$$= \frac{2GmMld \sin \theta}{(l^2 + d^2 - 2ld \cos \theta)^{\frac{3}{2}}}$$

If the torsional rigidity of the suspending fibre is again c, the restoring

couple due to the suspension is $c\varphi$ and the equilibrium position is given by:

$$c\varphi = \frac{2GmMld \sin \theta}{(l^2 + d^2 - 2ld \cos \theta)^{\frac{3}{2}}}$$

$$\text{or } G = \frac{c\varphi(l^2 + d^2 - 2ld \cos \theta)^{\frac{3}{2}}}{2mMld \sin \theta} \qquad . \quad . \quad . \quad (23)$$

In this calculation no allowance has been made for the cross-attractions between the spheres, although in the actual experiment this was done. Boys' result gave G as $6{\cdot}66 \times 10^{-8}$ c.g.s. units; the value accepted as correct today is $6{\cdot}67 \times 10^{-8}$ c.g.s. units.

(b) Bullion Balance Method of Measuring the Gravitational Constant

In 1893, Poynting managed to weigh on an ordinary balance the gravitational force between two spheres; he used a bullion balance, which is similar in construction and accuracy to a chemical balance but is built to carry very large loads. Two spheres, each weighing $\frac{1}{2}$ cwt, were suspended from the beam of a balance (Fig. 5.11), and

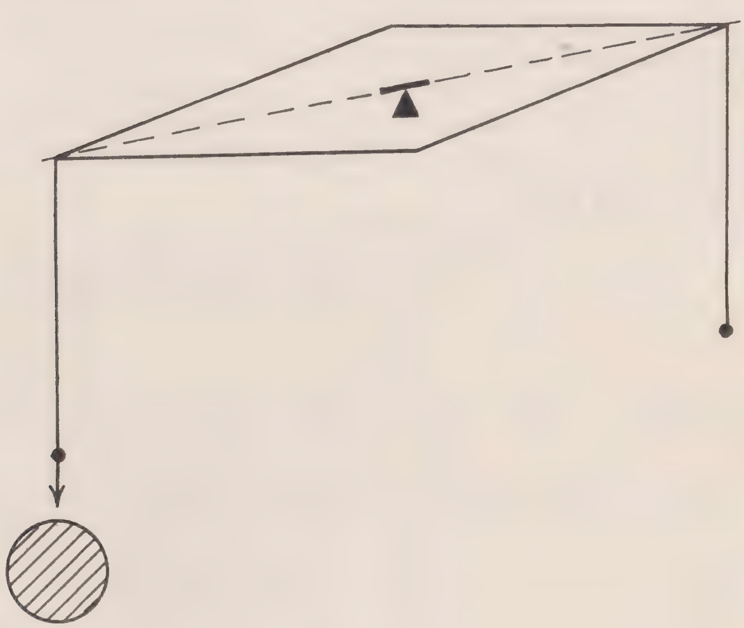


Fig. 5.11

under one of them was placed another sphere weighing over 3 cwt; the gravitational attraction of this sphere caused an apparent increase in weight on one side of the balance which was counteracted by weights placed on the other side.

If the suspended spheres are of mass m, and the large sphere of mass

M, the extra gravitational force acting on one suspended sphere is given by:

$$\text{force} = \frac{GmM}{d^2}$$

where d is the distance between the centres of the spheres.

If a small mass δm has to be added to restore the balance, then

$$\delta m \, . \, g = \frac{GmM}{d^2}$$

$$\text{or } G = \frac{d^2 \, g \, \delta m}{mM} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

In the apparatus used by Poynting, δm was of the order of 5 mg, which was well within the capabilities of the balance. Many corrections had to be applied before an accurate result was achieved. The cross-attraction between opposite spheres, attraction between the spheres and the beam of the balance, and tilting of the apparatus due to the 3-cwt mass were all eliminated by careful experimenting.

Example 2. *Two small spheres, each of radius* 2 *cm and mass* 350 *gm are suspended below the pans of a sensitive balance. A* 1-*mg rider is moved* 1/20*th of a division along the right-hand beam* (*increasing the effective load in the right-hand pan by* 1/200 *mg*), *and causes the pointer of the balance to be deflected* 10 *divisions.*
A large sphere of mass 40 *kg and radius* 10 *cm is then placed under the left-hand sphere, nearly touching it; what is the new deflection of the pointer? Where must the large sphere be placed to reduce the deflection to zero?* ($G = 6{\cdot}67 \times 10^{-8}$ *c.g.s. units*, $g = 980$ *cm.sec*$^{-2}$.)

The force between the spheres is given by:

$$\text{Force} = \frac{6{\cdot}67 \times 10^{-8} \times 350 \times 40 \times 10^3}{12^2} \text{ dynes}$$

$$= 0{\cdot}0065 \text{ dynes.}$$

Now $\frac{1}{200}$ mg has a weight of $\dfrac{980}{200 \times 1000} = 0{\cdot}0049$ dynes. This force causes a deflection of 10 divisions, thus 1 division represents 0·00049 dynes.

When the large sphere is in place, its gravitational force overrides the effect of the weight of the rider by (0·0065 − 0·0049) dynes, i.e. 0·0016 dynes, and this produces a deflection of $\dfrac{0{\cdot}0016}{0{\cdot}00049} = 3{\cdot}3$ divisions on the other side.

The deflection will be zero when the gravitational force of the large sphere is equal to the weight added by the rider: if this occurs when the distance between the centres of the spheres is d cm, then:

$$\frac{6{\cdot}67 \times 10^{-8} \times 350 \times 40 \times 10^3}{d^2} = 0{\cdot}0049$$

$$\text{or } d = \sqrt{\frac{6{\cdot}67 \times 10^{-8} \times 350 \times 40 \times 10^3}{0{\cdot}0049}} \text{ cm}$$

$$= 13{\cdot}8 \text{ cm.}$$

Therefore the large sphere must be lowered 1·8 cm.

(c) Mountain Method of Measuring the Gravitational Constant and 'Weighing the Earth'

The gravitational force on a small body of mass m at the surface of the Earth is given by:

$$f = \frac{GmE}{R^2}$$

where E is the mass of the Earth and R its radius. But the force on a body is its weight mg, thus:

$$mg = \frac{GmE}{R^2}$$

$$\text{or } G = \frac{gR^2}{E} \qquad . \quad . \quad . \quad . \quad . \qquad (25)$$

Now g is known from pendulum experiments and R is known from astronomical measurements, so that if E, the mass of the Earth, is measured, then G can be calculated or vice versa. The fact that such experiments lead to a value for the mass of the Earth resulted in them becoming known as 'Weighing the Earth'.

Experiments designed to weigh the Earth were performed as early as 1740 by Bouguer, working in the Andes, and in 1774 by Maskelyne,

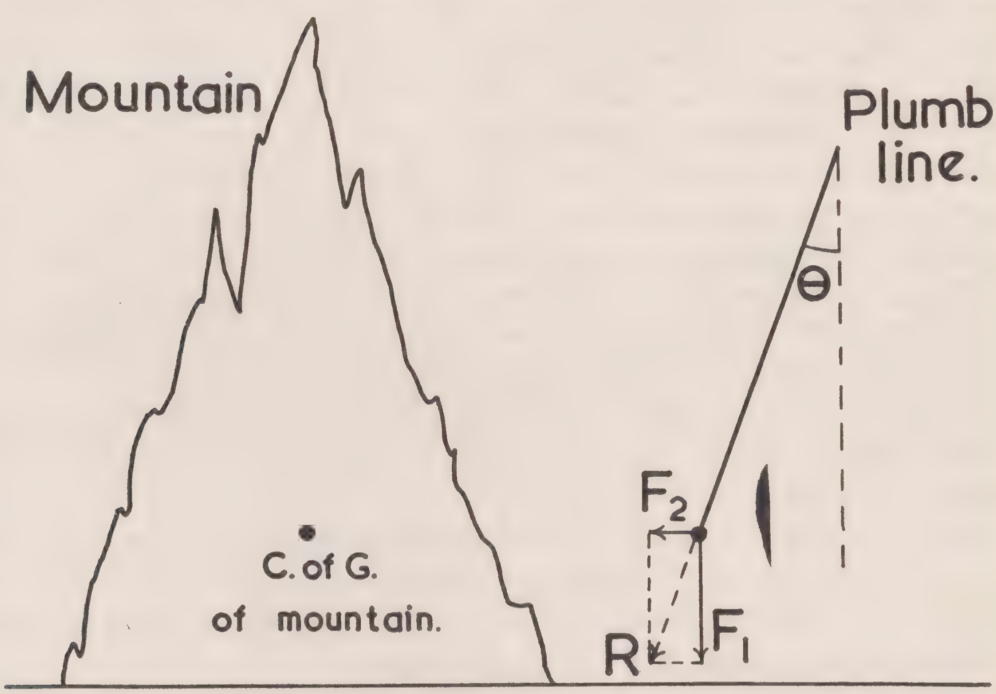


Fig. 5.12

who worked in Scotland. Both of these experimenters worked near a mountain (Schiehallion in Perthshire in the case of Maskelyne's experiment) the shape and density of which could be accurately surveyed; this meant that both the mass and the position of the centre of gravity of the mountain were known.

If a plumb-line is hung near this mountain (Fig. 5.12) the bob, of mass m, will experience a downward force due to the Earth's pull given by:

$$F_1 = \frac{GmE}{R^2},$$

and a sideways attraction given by:

$$F_2 = \frac{GmM}{d^2},$$

if M is the mass of the mountain and d the distance between its centre of gravity and the bob.

These two forces will produce a resultant force in a direction θ from the true vertical where:

$$\tan\theta = \frac{F_2}{F_1}$$

$$= \frac{GmM}{d^2} \cdot \frac{R^2}{GmE},$$

$$\text{or } E = \frac{MR^2}{d^2 \tan\theta} \qquad \text{. . . . . (26)}$$

The direction of the resultant force is the apparent direction of gravity at that point, and is also the direction in which the plumb-line will set. Bouguer measured θ by setting the zero of the angular scale of an astronomical telescope 'vertical' by means of the plumb-line and measuring the position of a fixed star; he then withdrew to a position many miles from the mountain where the plumb-line set truly vertical, re-erected his apparatus, using the same method, and measured the position of the same star. The difference between the two readings, after allowing for the curvature of the Earth's surface, gave the angle θ. It was only some six seconds of arc in his experiments!

Maskelyne used a similar method of measurement, but took readings on both sides of the mountain, hence the difference between the two readings gave 2θ.

Bouguer worked under great experimental difficulties and his result is not very reliable, but Maskelyne obtained the value $5{\cdot}5 \times 10^{27}$ gm, which is only about 8 per cent. lower than the best modern value for the mass of the Earth, i.e. $5{\cdot}98 \times 10^{27}$ gm.

5.11 Mass of a Celestial Body Derived from the Motion of its Satellites

The methods derived so far enable an estimate to be made of the mass of the Earth only and cannot be applied to other planets and stars; if such a celestial body possesses satellites, however, then its mass can be deduced from the motion of the satellites as follows.

Imagine a satellite of mass m revolving in a circular orbit of radius r around a central body of mass M.

The gravitational force between them is GmM/r^2 and hence the acceleration of the smaller body towards the larger will be GM/r^2. Since the smaller body is moving in a circular orbit, it must have a radial acceleration equal to v^2/r if v is the orbital velocity of the satellite. This acceleration is provided by the gravitational force, thus:

$$\frac{v^2}{r} = \frac{GM}{r^2} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

If t is the time for one revolution of the satellite, then

$$v = \frac{2\pi r}{t} \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and substituting this in Equation (27) gives:

$$\frac{4\pi^2 r}{t^2} = \frac{GM}{r^2} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

$$\text{or } \frac{r^3}{t^2} = \frac{GM}{4\pi^2} \quad . \quad . \quad . \quad . \quad . \quad (30)$$

This will be recognised as Kepler's third law, since $GM/4\pi^2$ is a constant for any particular system.

Rearranging Equation (30) gives:

$$M = \frac{4\pi^2}{G} \cdot \frac{r^3}{t^2},$$

and thus the mass of the Sun can be calculated from a knowledge of r^3/t^2 for each of its planets, Mars, Earth, Neptune, etc.; or the mass of Saturn from the motion of each of its 10 'moons'.

Applying this method to find the mass of the Sun, we have:

Planet	*Radius of Orbit*	*Time for One Revolution*	$\frac{r^3}{t^2}$
Mercury . . .	58×10^{11} cm	$0{\cdot}76 \times 10^7$ sec	$3{\cdot}38 \times 10^{24}$
Venus . . .	108	1·95	3·32
Earth . . .	150	3·15	3·36
Mars . . .	228	5·93	3·37
Jupiter . . .	780	37·4	3·38
Saturn . . .	1420	93·0	3·33
Uranus . . .	2860	264	3·36
Neptune . . .	4500	520	3·38
		Mean	$3{\cdot}36 \times 10^{24}$

Thus the mass of the Sun is:

$$M = \frac{4\pi^2}{6{\cdot}67 \times 10^{-8}} \times 3{\cdot}36 \times 10^{24} \text{ gm}$$
$$= 1{\cdot}99 \times 10^{33} \text{ gm.}$$

Example 3. *The Moon revolves in a circular orbit of radius* 384 $\times$ 10^3 *km about the Earth and encircles it* 13 *times in one year. Find the radius of the Earth if the gravitational acceleration at its surface is* 981 *cm.sec*$^{-2}$.

The gravitation force acting on the Moon is GmM/a^2, where G is the gravitational constant, m the mass of the moon, M the mass of the Earth and a the radius of the Moon's orbit.

This must provide the centripetal force on the Moon, i.e. mv^2/a, where v is the Moon's orbital velocity.

$$\text{Thus } \frac{GmM}{a^2} = \frac{mv^2}{a}$$

$$\text{or } v^2 = \frac{GM}{a}.$$

$$\text{But } v = \frac{2\pi a}{T} \text{ where } T \text{ is the Moon's periodic time,}$$

$$\text{thus } \frac{4\pi^2 a^3}{T^2} = GM.$$

But if R is the radius of the Earth and g the gravitational acceleration at its surface, then:

$$g = \frac{GM}{R^2}$$

$$\text{or } gR^2 = GM,$$

$$\text{thus } \frac{4\pi^2 a^3}{T^2} = gR^2,$$

$$\text{whence } R = \frac{2\pi a}{T}\sqrt{\frac{a}{g}}.$$

Taking 1 year as $365\frac{1}{4}$ days, we have:

$$T = \frac{1461 \times 24 \times 60 \times 60}{4 \times 13} \text{ sec,}$$
$$= 2{\cdot}43 \times 10^6 \text{ sec;}$$

$$\text{thus } R = \frac{2\pi \,.\, 384 \times 10^8}{2{\cdot}43 \times 10^6}\sqrt{\frac{384 \times 10^8}{981}} \text{ cm}$$
$$= 6{\cdot}22 \times 10^8 \text{ cm.}$$

5.12 Accurate Measurement of the Acceleration due to Gravity

In some of the methods discussed above for the measurement of the gravitational constant or the mass of the Earth, a knowledge of the value of g is needed before the final calculation can be made.

The simple pendulum method of measuring g is of low accuracy for several reasons. Its theory is incomplete due to the approximation introduced when finding the moment of inertia of the bob (see page 104), and although the pendulum is far from rigid, the theory fails to allow for relative motion between the various parts.

These defects can be overcome by the use of a solid or *Compound Pendulum*, and by writing out a more complete theory for its motion. Let the pendulum be a rigid body of any shape (Fig. 5.13) pivoted at a point P, distant l from its centre of gravity O. If the pendulum is dis-

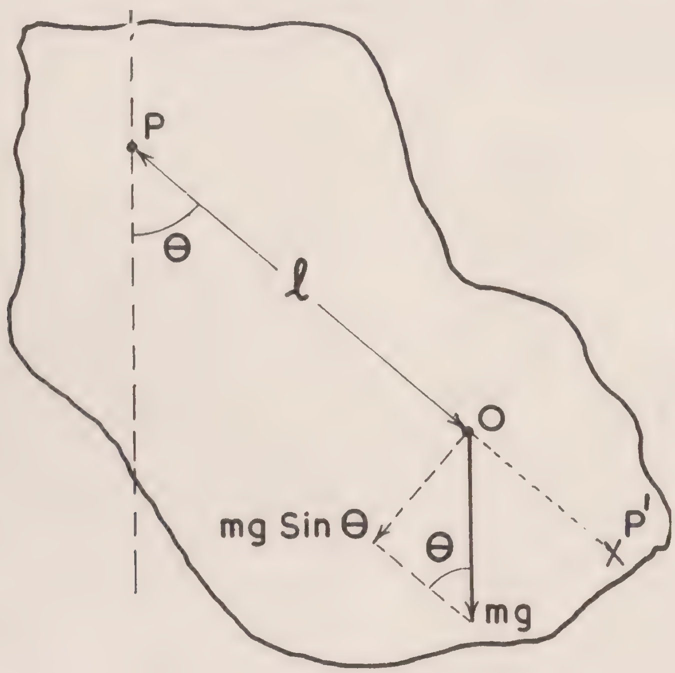


Fig. 5.13

placed by an angle θ, then the restoring force is $mg.\sin\theta$ and the restoring couple is $mgl\sin\theta$; but if the displacement is small, then $\sin\theta \simeq \theta$, thus:

$$\text{restoring couple} \simeq mgl\theta.$$

The restoring couple is therefore proportional to the displacement and thus the motion of the pendulum is simple harmonic.

For a simple harmonic system the periodic time is given by:

$$T = 2\pi\sqrt{\frac{\text{Moment of Inertia}}{\text{Restoring Couple at Unit Angular Displacement}}}$$

If k is the radius of gyration of the rigid body, then the moment of inertia about its centre of gravity is mk^2 and about the pivot (mk^2+ml^2); also the restoring couple at unit displacement (i.e. when $\theta = 1$) is mgl, thus:

$$T = 2\pi\sqrt{\frac{m(k^2+l^2)}{mgl}}$$

$$= 2\pi\sqrt{\frac{k^2+l^2}{gl}} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Rewriting this equation gives:

$$l^2 - \left(\frac{gT^2}{4\pi^2}\right)l + k^2 = 0 \quad . \quad . \quad . \quad . \quad (32)$$

which is a quadratic in l and thus has, in general, two roots; hence for every value of T there will be two values of l which satisfy the equation. This means that it should be possible to find two pivot points on the pendulum giving the same time of swing. If a graph is drawn of T against l we get the curve shown in Fig. 5.14 from which it is seen that two points of suspension at distances l_1 and l_2 from the centre of gravity give a time of swing T_0.

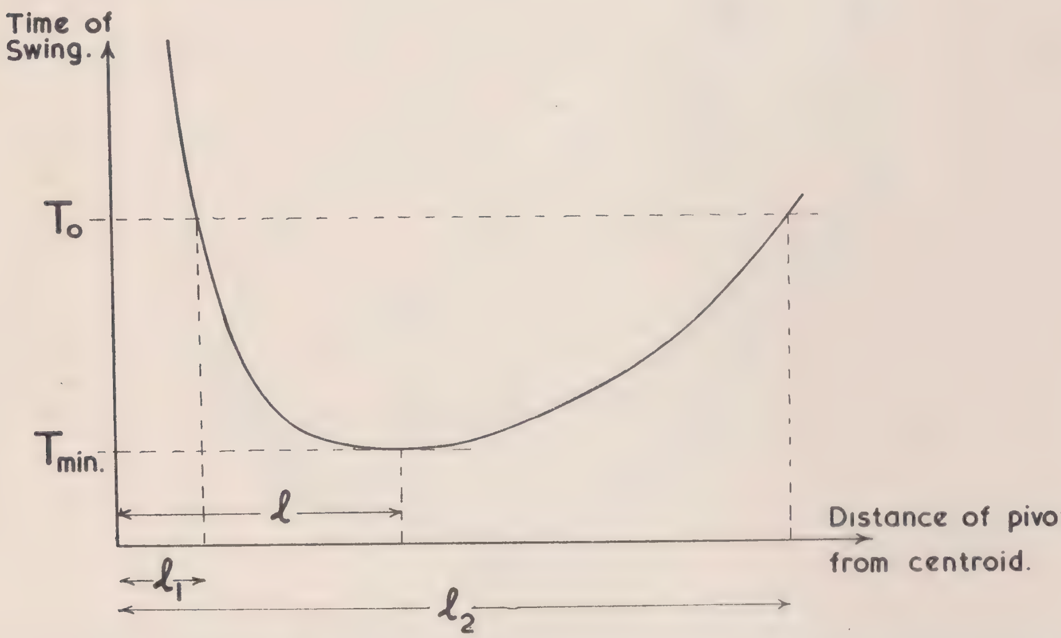


Fig. 5.14

It will also be noticed from the figure that there is a minimum time of swing $T_{\min}$; it can easily be shown that this occurs for a point of suspension at $l = k$, thus $T_{\min} = 2\pi\sqrt{2k/g}$.

Now l_1 and l_2 will be the roots of Equation (32) when $T = T_0$; we know that the sum of the roots of a quadratic equation is equal to the coefficient of the term in x, with its sign changed; thus, from Equation (32),

$$l_1 + l_2 = \frac{gT_0^2}{4\pi^2}$$

$$\text{or } T_0 = 2\pi\sqrt{\frac{l_1 + l_2}{g}} \qquad . \quad . \quad . \quad . \quad (33)$$

Alternatively we could produce this result as follows. Equation (31) gives:

$$T = 2\pi\sqrt{\frac{k^2 + l^2}{gl}},$$

but this cannot be used to find g, for although we can measure T and l, it would be very difficult to find k for the irregular body shown in Fig. 5.13. It will be seen, however, from the graph of T against l, Fig. 5.14,

that the same time of swing T_0 can be obtained at two points of suspension, distant l_1 and l_2 from the centre of gravity respectively. Substituting these values in Equation (31) gives two equations from which k can be eliminated, thus:

$$T_0 = 2\pi\sqrt{\frac{k^2 + l_1^2}{gl_1}}$$

$$\text{and } T_0 = 2\pi\sqrt{\frac{k^2 + l_2^2}{gl_2}}$$

$$\text{Therefore } \frac{gl_1T_0^2}{4\pi^2} = k^2 + l_1^2$$

$$\text{and } \frac{gl_2T_0^2}{4\pi^2} = k^2 + l_2^2.$$

Subtracting the latter equation from the former gives:

$$\frac{gT_0^2}{4\pi^2}(l_1 - l_2) = l_1^2 - l_2^2,$$

$$\text{thus } \frac{gT_0^2}{4\pi^2} = l_1 + l_2,$$

$$\text{and } T_0 = 2\pi\sqrt{\frac{l_1 + l_2}{g}}, \text{ as before.}$$

Throughout this work there has been no restriction on the direction of the line joining the pivot and the centroid; in fact, the time of swing is given by $T_0 = 2\pi\sqrt{\frac{k^2 + l_1^2}{gl}}$ if the body is pivoted at any point on the circle of radius l_1 around the centre of gravity, and similarly for a circle of radius l_2 (Fig. 5.15).

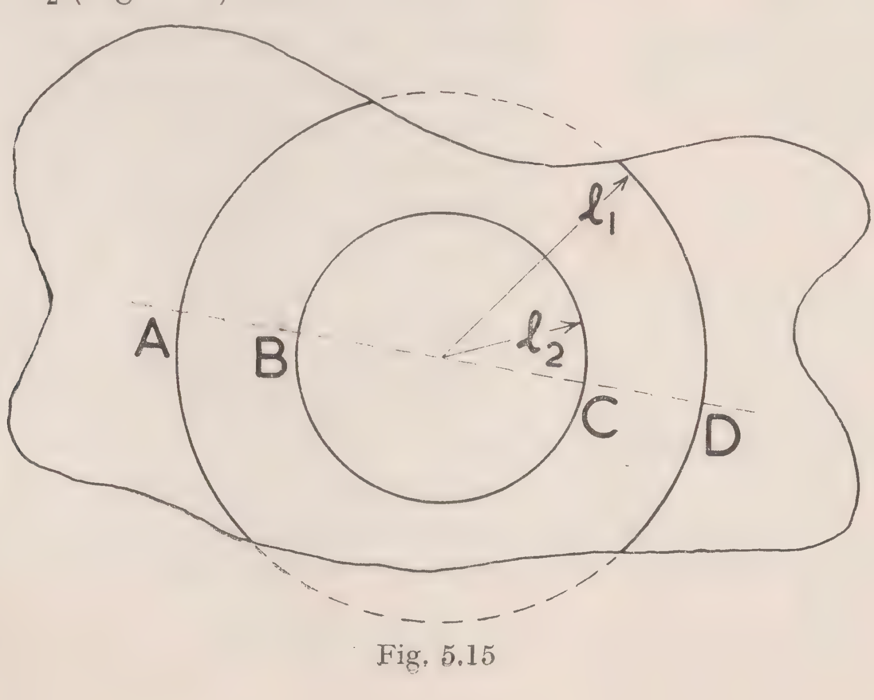


Fig. 5.15

If a straight line is drawn through the centre of gravity, it will cut these circles in the four points $ABCD$, and the distance between either of the pairs AC, BD is equal to $(l_1 + l_2)$.

In order to find an accurate value of g, then, all that has to be done is to identify two points lying on a straight line through the centre of gravity of the body and about which it has the same time of swing. These points must be asymmetrically disposed with respect to the centroid, i.e. the pairs AD or BC must not be chosen. The distance between these points is then measured and the value inserted in Equation (33), thus enabling g to be calculated if T_0 is known.

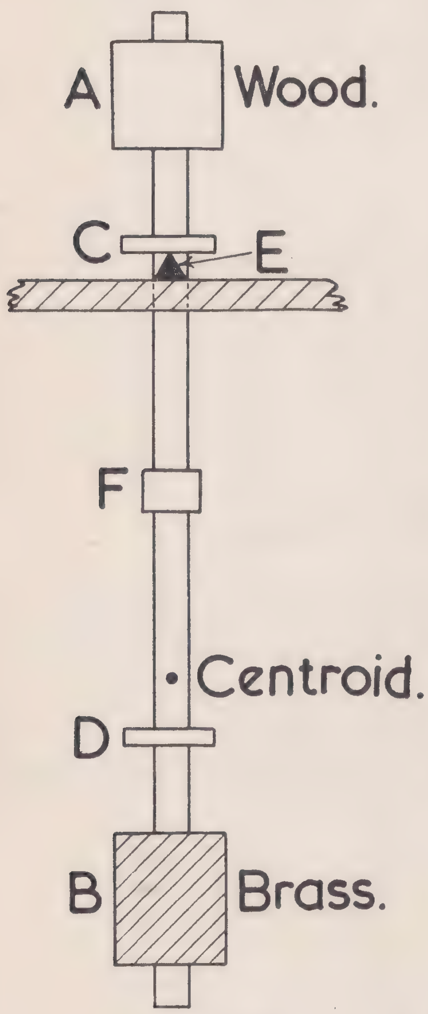


Fig. 5.16

To avoid the tedium of locating a pair of suitable pivot points on a complex body, a simpler form of pendulum, invented by Kater and named after him, is used. This consists of a long rod (Fig. 5.16) having two large bobs A and B placed symmetrically at the ends; these are identical in shape and size, but one is of metal and the other of wood; the centre of gravity is thus much nearer to one end than the other. Two optically flat surfaces are fixed symmetrically to the rod at C and D, and the pendulum pivots about a knife-edge E. The bobs A,B and the planes C,D can be moved along the rod, but are kept in roughly symmetrical positions. They are adjusted until the time of swing of the pendulum is the same, whether it is supported on plane C or turned up side down and supported on the plane D. The final adjustment to equality is made with the small bob F at the centre of the rod. The distance between the two planes is then equal to the distance $(l_1 + l_2)$. The asymmetrical position of the centroid ensures that the correct pair of pivot points (A,C or B,D of Fig. 5.15) is selected. A pendulum which is used in this fashion, i.e. first one way up and then the other, is described as a *Reversible Pendulum.*

The compound pendulum removes the two big faults of the simple pendulum and proves to be a method of such accuracy that a number of small corrections have to be made if full advantage of its precision is to be gained.

Air resistance and sway of the knife-edge as the pendulum swings

cannot be eliminated completely and so small allowances must be made. When swinging through a large amplitude, the period of a pendulum is slightly longer than when its swing is small. This is because the restoring couple is really proportional to sin θ and not θ as has been assumed in the theory. Once again, however, the correction for this can be derived; it is:

$$T_0 = T_m\left(1 - \frac{\theta^2}{16}\right)$$

where T_m is the period measured when the pendulum is swinging through θ radians and T_0 is the period that it would have if the restoring couple were proportional to θ.

Since we have assumed this condition in the theory, the corrected value T_0 must be substituted in Equation (33) and not the measured value T_m.

The method of measuring T_m when doing this experiment to the highest accuracy is interesting. The Royal Observatory will supply electrical impulses at one-second intervals derived from a standard clock (rather similar to the B.B.C. 'pips'), and will distribute them over the country either by telephone line or by radio on a special wavelength. The pendulum is then adjusted so that it swings in step with these pips over a long period of time. With no further aid than watching the pendulum and listening to the pips, a lack of agreement of a quarter period can easily be detected between the two. Hence, if the two can be made to agree to this extent over a period of an hour, the pendulum makes $3600 \pm \frac{1}{4}$ oscillations in 3600 seconds, thus its period is known to be 1 second to better than 1 part in 10,000.

In practice, a degree of accuracy rather higher than this can be achieved, and it is this, together with the accuracy with which the length of the pendulum can be measured, that makes the method capable of giving results of such high precision.

Example 4. *A flat circular plate of radius* 10 *cm is hung from a point on its rim with its plane vertical. Find the period of small oscillations and also the position of a pivot point, not on the rim, which will give the same period.* (g = 981 *cm.sec*$^{-2}$.)

The plate is being used as a compound pendulum, for which the period is given by:

$$T = 2\pi\sqrt{\frac{k^2 + l^2}{gl}},$$

where k = radius of gyration

l = distance from centroid to pivot.

If the radius of the plate is r, then $k^2 = r^2/2$ and $l = r$,

$$\text{thus } T = 2\pi\sqrt{\frac{(r^2/2) + r^2}{gr}}$$

$$= 2\pi\sqrt{\frac{3r}{2g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Substituting numerical values gives:

$$T = 2\pi\sqrt{\frac{3 \times 10}{2 \times 981}} \text{ sec}$$
$$= 0{\cdot}78 \text{ sec.}$$

If there is another pivot point, at a distance l' from the centroid giving the same time of swing, then:

$$T = 2\pi\sqrt{\frac{k^2 + l'^2}{gl'}},$$

$$\text{thus } T^2 = \frac{4\pi^2(k^2 + l'^2)}{gl'},$$

or $4\pi^2 l'^2 - T^2 gl' + 4\pi^2 k^2 = 0$.

But $k^2 = r^2/2$ and $T^2 = 6\pi^2 r/g$ from Equation (a) above, thus

$4\pi^2 l'^2 - 6\pi^2 rl' + 2\pi^2 r^2 = 0$.

Solving this for l' gives:

$$l' = \frac{6\pi^2 r \pm \sqrt{36\pi^4 r^2 - 32\pi^4 r^2}}{8\pi^2},$$

$$\text{whence } l' = \frac{6\pi^2 r \pm 2\pi^2 r}{8\pi^2},$$

$$\text{or } l' = r \text{ or } r/2.$$

The point given by $l' = r/2$ is not on the rim, hence the same period can be achieved by pivoting the plate 5 cm from the centre.

5.13 Variations in Gravity near the Surface of the Earth

Many variations in the acceleration due to gravity occur over the surface of the Earth and most of these are due to local irregularities in the surface layers, but there are two systematic variations, those caused by distance from sea-level and by latitude, that can be investigated theoretically.

(a) Variation of Gravity with Latitude

The Earth is not truly spherical in form, but bulges at the equator, this change in the radius of the Earth with latitude produces a variation in g; also the centrifugal effect, which acts on a body on the surface of the Earth and so modifies the gravitational attraction, varies with latitude.

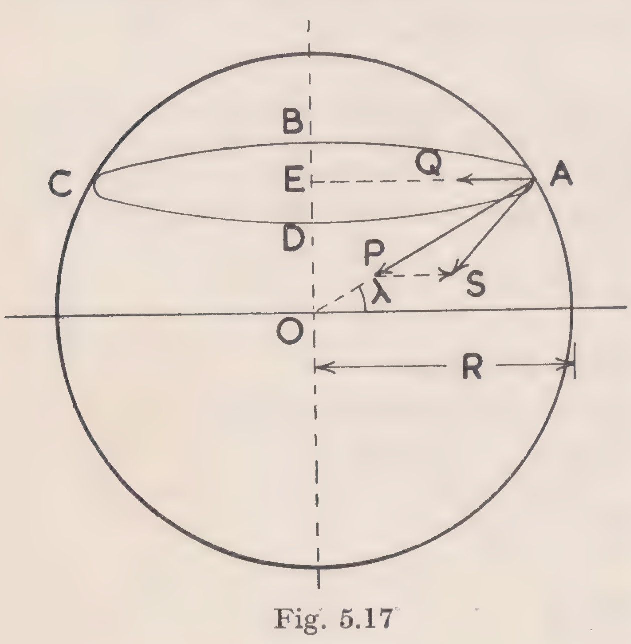


Fig. 5.17

If a body of mass m is situated at a point A, latitude λ, on the surface of the Earth (Fig. 5.17), then the gravitational attraction on the body is of magnitude mg acting along AO, and is represented by the vector

AP. The symbol g indicates the value due to gravity which would exist at A if the Earth were spherical and in the absence of any disturbing effect; it is therefore equal to $\frac{GE}{R^2}$ (see Equation (27)). Due to the rotation of the Earth, the body at A moves around the circle $ABCD$ of radius AE; to make the body move in this circle, a centripetal force $m.AE.\omega^2$ must be exerted on it where ω is the angular velocity of rotation of the Earth, i.e. 2π radians per day. This force acts along AE and is represented by the vector AQ.

Now this centripetal force can be provided only by a component of the gravitational effect of the Earth, and hence any acceleration of the body in the downward direction—'falling' in our sense—will be provided by the component of the gravitational force that is left after it has provided the centripetal force; thus the apparent gravitational force at A will be the *difference* of the two vectors AP and AQ. The subtraction can be done by drawing a vector PS equal to AQ but in the opposite direction and then adding it to the vector AP. The vector AS then represents the apparent gravitational effect on a body at A and shows the direction in which a plumb-line would set, although this direction does not pass through the centre of the Earth. The direction of the plumb-line is, however, perpendicular to the actual surface of the Earth, since when the Earth was cooling but still fluid, any component of the apparent gravitational force acting along the surface would cause a flow of the surface in the same direction, and this process would continue until the surface of the Earth had everywhere set itself perpendicular to the apparent gravitational force. This results in the Earth taking up a spheroidal shape, flattened at the poles and bulging at the equator, as mentioned above.

The magnitude of the apparent gravitational acceleration, allowing only for the centrifugal effect and not for the ellipticity of the Earth, can be calculated as follows.

The vector AS is given by:

$$AS^2 = AP^2 + PS^2 - 2AP.PS.\cos\lambda$$

and writing

$$AS = mg_\lambda$$

where g_λ is the gravitational acceleration of a body at A, leads to:

$$(mg_\lambda)^2 = (mg)^2 + (m\omega^2 R\cos\lambda)^2 - 2mg\,.\,m\omega^2 R\cos^2\lambda,$$

since $AE = R\cos\lambda$.

$$\text{Thus } g_\lambda{}^2 = g^2 + \omega^4 R^2\cos^2\lambda - 2g\omega^2 R\cos^2\lambda \quad . \quad . \quad . \quad (34)$$

Now g is the value of gravitational acceleration which would exist if the Earth were spherical and no centrifugal effect existed; it is therefore a quantity which cannot be measured directly and must be removed

from the equation. This is done as follows. At the equator, when $\lambda = 0$:

$$(g_\lambda)^2 = g^2 + \omega^4 R^2 - 2g\omega^2 R$$
$$= (g - \omega^2 R)^2$$

and if g_E is written for this value of g_λ, then

$$g_E = g - \omega^2 R$$
$$\text{or } g = g_E + \omega^2 R.$$

Substituting this value in Equation (34) gives:

$$g_\lambda^2 = (g_E + \omega^2 R)^2 + \omega^4 R^2 (\cos^2 \lambda) - 2(g_E + \omega^2 R)\omega^2 R \cos^2 \lambda,$$

which, after some rearrangement, becomes:

$$g_\lambda^2 = g_E^2 + \omega^2 R(2g_E - \omega^2 R) \sin^2 \lambda \quad . \quad . \quad . \quad . \quad (35)$$

This formula relates gravity at any latitude to the value obtained at the equator and it can be shown that if the ellipticity of the Earth is taken into account, i.e. if R varies with latitude as well, then Equation (35) becomes:

$$g_\lambda = g_E(1 + k \sin^2 \lambda) \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

where k is a constant.

In 1903, Helmert, using data collected from all inhabited latitudes, gave the equation as:

$$g_\lambda = 978{\cdot}00\ (1 + 0{\cdot}005310 \sin^2 \lambda) \quad . \quad . \quad . \quad . \quad (37)$$

(b) Variation of Gravity with Height above Sea-level

There are two ways in which a body may find itself above sea-level, it may either be placed on a prominence of the Earth's surface, as for example at the top of a mountain (P' in Fig. 5.18), or it may be removed altogether from the surface of the Earth, as an aeroplane in flight (P in Fig. 5.18). In either case, the increased distance to the centre of the Earth causes a change in the gravitational force, but the treatment of the two cases is rather different.

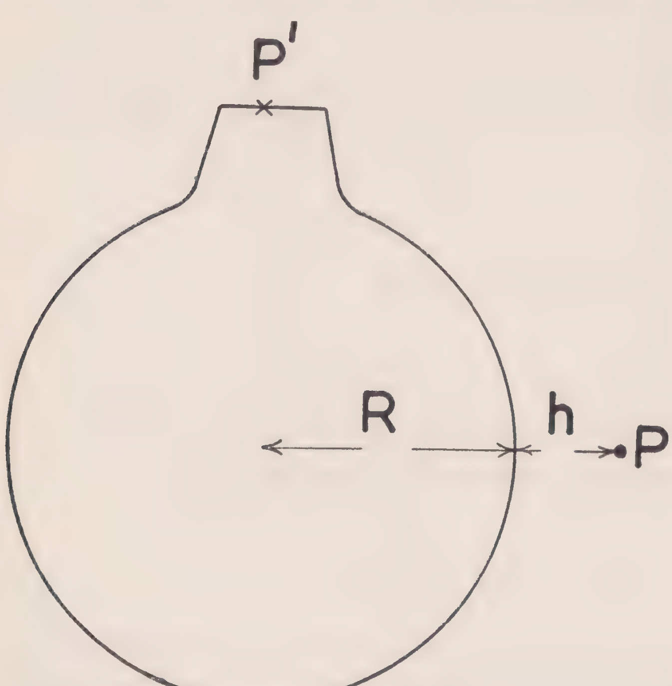


Fig. 5.18

Consider first of all the body removed from the surface at P.

At a height h above sea-level. the attraction

due to the Earth is due to the whole mass of the Earth, but the distance to its centre is $(R + h)$; substituting these values in Equation (27) gives:

$$g_h = \frac{GE}{(R+h)^2},$$

where g_h is the value of the gravitational acceleration at a height h.

$$\text{Thus } g_h = \frac{GE}{R^2\left(1+\frac{h}{R}\right)^2}$$

But GE/R^2 is the value of the gravitation acceleration at the surface of the Earth, and may be written as g_s,

$$\text{or } g_h = g_s\left(1 + \frac{h}{R}\right)^{-2}$$

Now h/R is very small; therefore, expanding the term $(1 + h/R)^{-2}$ by the binomial theorem and ignoring terms in $(h/R)^2$ and higher orders, gives:

$$g_h = g_s\left(1 - \frac{2h}{R}\right)$$

$$\text{or } g_s - g_h = g_s \cdot \frac{2h}{R},$$

which can be written as

$$\Delta g = g_s \cdot 2h/R \qquad . \quad . \quad . \quad . \quad (38)$$

where Δg is the decrease in gravitational acceleration occurring over a height h.

At a point such as P' on the top of a mountain, allowance should be made for the extra gravitational attraction of the mass of the mountain. Bouguer suggested that the correct formula should be:

$$g_h = g_s\left(1 - \frac{2h}{R} + \frac{3hd}{2RD}\right) \qquad . \quad . \quad (39)$$

where d is the mean density of the mountain and D is the mean density of the whole Earth.

Either of the Equations (38) or (39) indicates that the gravitational force decreases with height and this raises the interesting possibility of a body being projected sufficiently high to become free from the Earth's gravitational effect. We can investigate the necessary conditions as follows.

The gravitational force on a body of mass m at a distance r from the centre of the Earth (where $r > R$) is given by:

$$\text{Force} = \frac{GmE}{r^2}.$$

If the body is moved a further distance δr away from the Earth, then

the work δW necessary to make this move is given by (force $\times$ distance moved), thus:

$$\delta W = \frac{GmE\ \delta r}{r^2}.$$

The work needed to remove the body entirely from the Earth is the sum of all these elements from $r = R$ to $r = \infty$, thus the total work W is given by:

$$\begin{aligned} W &= \int_R^\infty \frac{GmE\ dr}{r^2} \\ &= GmE \int_R^\infty \frac{dr}{r^2} \\ &= GmE \left[\frac{-1}{r} \right]_R^\infty \\ &= \frac{GmE}{R}. \end{aligned}$$

If the body is to escape from the Earth, it must start on its travels with enough energy to be able to do all of this work; for example, if it is a shell fired from a gun, then its initial velocity must be such that its kinetic energy is equal to GmE/R, thus:

$$\tfrac{1}{2}mv^2 = \frac{GmE}{R}$$

$$\text{or } v = \sqrt{\frac{2GE}{R}} \quad . \quad . \quad . \quad . \quad . \quad (40)$$

This particular value is known as the *Escape Velocity*, since it is the velocity with which a projectile must be launched (in any direction) in order to escape from the Earth; its value is $1{\cdot}1 \times 10^6$ cm. sec^{-1} or about 25,000 mph and this high value constitutes one of the obstacles to interplanetry travel.

Every heavenly body has its own value of escape velocity, obtained by inserting the appropriate values for E and R in Equation (40). For the Earth, the escape velocity is nearly 20 times greater than the average velocity of the molecules in the atmosphere, consequently we are in no great danger of our air vanishing into space. The escape velocity for the Moon is 2×10^5 cm.sec^{-1}; this is only four times greater than the average velocity of oxygen molecules. Consequently those oxygen molecules which from time to time have acquired velocities much above the average have been able to escape, and the Moon now has no atmosphere as we understand the term.

Example 5. *The aneroid barometer in an aeroplane flying at* 10,000 *ft reads* 65·00 *cm of mercury. What would a Fortin barometer read at the same height? Take the radius of the Earth as* 4000 *miles.*

An aneroid barometer measures the atmospheric pressure and scales it into cm of mercury by means of the equation:

$$\text{Pressure} = g\varrho h \text{ dyne.cm}^{-2},$$

where g is the standard value of the gravitational acceleration at the surface of the Earth, ϱ the density of mercury and h the 'height of the barometer' all in c.g.s. units.

The Fortin barometer balances the hydrostatic pressure produced by a column of mercury against the atmospheric pressure. Now at a distance H above the surface of the Earth, the gravitational acceleration is given by $g_H = g\left(1 - \frac{2H}{R}\right)$, where R is the radius of the Earth: thus the hydrostatic pressure produced by a mercury column h' cm high is given by:

$$\text{Pressure} = g_H\varrho h'$$

(i.e. g_H see)

$$= g\left(1 - \frac{2H}{R}\right)\varrho h',$$

and since both barometers are measuring the same pressure, this value can be equated with the one above, thus:

$$g\left(1 - \frac{2H}{R}\right)\varrho h' = g\varrho h$$

$$\text{or } h' = \frac{h}{\left(1 - \frac{2H}{R}\right)}$$

$$= h\left(1 + \frac{2H}{R}\right), \text{ since } \frac{2H}{R} \text{ is small.}$$

$$\text{Thus } h' = 65\left(1 + \frac{2 \times 10{,}000}{4000 \times 5280}\right)$$

$$= 65{\cdot}00 \times 1{\cdot}00095$$

$$= 65{\cdot}06 \text{ cm.}$$

Summary of Units derived in this Chapter

The only new unit derived in this chapter is the Universal Gravitational Constant, G; which, although it is a constant, has dimensions $[M^{-1}L^3T^{-2}]$. It is thus measured in gm^{-1}.cm^3.sec^{-2} in the c.g.s. system (value $6{\cdot}67 \times 10^{-8}$), kg^{-1}.m^3.sec^{-2} in the M.K.S. system (value $6{\cdot}67 \times 10^{-11}$) and lb^{-1}.ft^3.sec^{-2} in the f.p.s. system (value $1{\cdot}07 \times 10^{-9}$).

EXERCISES 5

1. Prove that a uniform spherical shell of matter exerts no gravitational attraction in its interior. Give an expression for the gravitational attraction due to the shell at an external point.

   Assuming that the earth is built up of concentric shells, each of uniform density, show that it is possible to determine the gravitational constant (G) and the value of the earth's mean density (Δ) by pendu-

lum experiments carried out at the top and bottom of a deep mine-shaft.

A mass of 5 kilograms is weighed on a balance at the top of a tower 20 metres high. The mass is then suspended from the pan of the balance by a fine wire 20 metres long and is reweighed. Find the change in weight in milligrams. Assume that the radius of the earth is 6,330 kilometres. (Northern Univ. H.S.C. Schol.)

2. Suppose a straight smooth tunnel to connect any two points on the surface of the earth (assumed to be of uniform density throughout). Find the time taken for a body starting at rest at one end of the tunnel to reach the other end. The radius of the earth is 4,000 miles. (Manchester Univ. Schol.)

3. Describe Cavendish's method of determining the gravitational constant G. Point out carefully the sources of error in the experiment. What are the dimensions of G?

Two lead spheres each of diameter 1 metre are placed with their centres 2 metres apart. If they are initially at rest, find the relative velocity with which they will come into contact, assuming that the only force acting on them is that of their mutual gravitational attraction.

($G = 6{\cdot}7 \times 10^{-8}$ c.g.s. units; density of lead $= 11{\cdot}3$ gm./cm.^{3})

(Cambridge G.C.E. Schol. level.)

4. How has the gravitational constant been measured in the laboratory?

The diameters of the planets Uranus and Mercury are nearly in the ratio 1/10. The acceleration due to gravity on their surface is approximately in the ratio 1/2. Obtain an approximate value for the ratio of their average densities. (Oxford Univ. Schol.)

5. A light rigid horizontal beam of length $2L$ carries at each end a small sphere of mass m, and is suspended at its mid-point by a fibre of torsional contant c. Find an expression for the period of small oscillations.

Two fixed spheres each of mass M are placed symmetrically in the equilibrium line of the beam, their centres being $2d$ apart ($d > L$). Find an expression for the new period of small oscillations. (The gravitational attraction between a fixed sphere and the small sphere furthest away from it may be neglected.) (Oxford Univ. Schol.)

6. Derive an expression for the acceleration of a particle moving along a circular path with uniform speed.

Show how this expression, combined with the law of gravitation, leads to the relationship $\omega^2R^3 = gr^2$ where ω is the mean angular velocity of the moon around the earth, R is the mean radius of the moon's orbit, and r is the radius of the earth.

Given that $R = 60r$, and that the period of the moon's rotation around the earth is $27\frac{1}{2}$ days, find R. (Cambridge H.S.C.)

7. State Newton's law of gravitation, and deduce the dimensions of the gravitational constant G.

Assuming that the planets move in circular orbits, show that the

square of the period of rotation of a planet about the sun is proportional to the cube of the radius of its orbit.

Given that the radius of the earth's orbit is $1{\cdot}5 \times 10^8$ km., and that the mass of the sun is 2×10^{30} kgm., deduce the value of G.

(Cambridge G.C.E. Advanced level.)

8. State the law of gravitational attraction and hence define the gravitational constant (G). How is this constant related to the acceleration (g) due to gravity? Describe a method of measuring G.

The ratio of the mass of the moon to that of the earth is as 1 : 81. If the radius of the moon is assumed to be one-fourth of that of the earth, what is the value of the acceleration due to gravity on the surface of the moon in terms of that at the earth's surface?

(London Univ. G.C.E. Schol. level.)

9. Assuming that the planets are moving in circular orbits, apply Kepler's laws to show that the acceleration of a planet is inversely proportional to the square of its distance from the sun. Explain the significance of this and show clearly how it leads to Newton's law of universal gravitation.

Obtain the value of g from the motion of the moon, assuming that its period of rotation round the earth is 27 days 8 hours and that the radius of its orbit is 60·1 times the radius of the earth.

Radius of earth = $6{\cdot}36 \times 10^6$ metres.

(Northern Univ. H.S.C. Schol. level.)

10. Describe a method for determining the gravitational constant G and show how it can be used to determine the mean density of the earth. Indicate briefly how any other data required are determined.

Using the data in the table compare (*a*) the radii of the orbits (assumed to be circular) of Mars and the earth, (*b*) their mean densities.

	Earth	*Mars*
'g' cm.sec.$^{-2}$ . . .	981	377
Radius (km.) . . .	6,400	3,400
Period (days) . . .	365·3	687·0

(Northern Univ. G.C.E. Schol. level.)

11. For a body moving with a constant acceleration, prove the equations relating (*a*) velocity and distance travelled, (*b*) distance travelled and time. How would you test *one* of these equations experimentally?

At what rate must the earth rotate for a body just to fly off the surface at the equator?

(Radius of the earth = 6400 km.)

(Cambridge G.C.E. Advanced level.)

12. State Newton's law of gravitation.

Show that, if the moon describes an approximately circular orbit of radius $3{\cdot}85 \times 10^{10}$ cm. in 27 days 7 hours, the acceleration due to gravity at the surface of the earth is consistent with the acceleration of the moon towards the earth (radius $6{\cdot}37 \times 10^8$ cm.)

Draw a diagram of apparatus suitable for determining in a laboratory the gravitational constant, G. State the quantities which must be measured, and show how to calculate the value of G.

(Northern Univ. H.S.C.)

13. Define *gravitational constant* and *gravitational acceleration* and obtain a relation between them.

Calculate the radius of the orbit (assumed circular) which the moon describes round the earth, assuming the time period for the orbit to be 27 days and the radius of the earth to be $6{\cdot}4 \times 10^8$ cm.

(Northern Univ. H.S.C.)

14. State Kepler's three laws of planetary motion. Taking the earth's period as unity (sidereal year) and its distance from the sun as unity (neglect the eccentricity of the earth's orbit), calculate

(*a*) The semi-major axis of the orbit of Mars, given its period as 1·88 sidereal years.

(*b*) The period of Pluto, given that its greatest and least distances from the sun are 49·49 and 29·55 respectively.

(Manchester Univ. Schol.)

15. Explain the use of the compound pendulum to measure the acceleration due to gravity, and describe how you would carry out such an experiment in the laboratory.

Assuming that the earth is a homogeneous sphere of radius 4×10^3 miles, find the percentage change in the period of a pendulum that should occur when it is taken down a mine 2,000 ft. deep.

(Oxford H.S.C.)

16. Discuss the use of the concept of Moment of Inertia. Find the ratio of the periods of oscillation of a long thin uniform rod suspended from one end and of a simple pendulum of the same length.

(Oxford Univ. Schol.)

17. A flat plate of irregular shape is pierced by a number of small holes, distributed at random, through which a knitting needle can pass easily. Describe how, using the plate and the needle, you would find the acceleration due to gravity and give the theory of the method.

A thin uniform rod swings as a pendulum about a horizontal axis at one end, the periodic time being 1·65 sec. If the mass of the rod is 125 gm., what is (*a*) its length, (*b*) its moment of inertia about the horizontal axis? (Northern Univ. G.C.E. Schol. level.)

18. It is desired to ascertain to an accuracy of 1 part in 1,000 the time period of a pendulum which is known to be very nearly two seconds. Describe the procedure if a standard pendulum clock with a period of exactly two seconds is available.

The time of oscillation of a thin uniform rod about one end is increased by 10 per cent. when a massive particle is attached to the other end. Compare the masses of the rod and the particle.

(It may be assumed that the time period of a body oscillating about an axis is given by $T = 2\pi\ (I/Mgh)^{\frac{1}{2}}$, where I is the moment of inertia of the body, mass M, about the axis, and h is the distance

between the axis and the centre of mass of the body. Proofs of any other formulæ used in the calculation should be given.)

(Northern Univ. G.C.E. Schol. level.)

19. Assuming the earth to be perfectly spherical, what is the angle between a plumb-line and the direction of the earth's radius at Cambridge (Lat. 52° N.)?

(1 degree of latitude = 110 km.)

(Cambridge Univ. Schol., King's College Group (Part).)

20. Describe an experiment by which the mean density of the earth has been determined, and explain how the result is calculated from the observations.

The variation of g at different places at sea has been studied by comparing the height of a mercury barometer with the corresponding readings of an aneroid barometer. If the difference can be read to the nearest 0·1 mm. of mercury, estimate the smallest change in g that can be detected at sea-level.

Discuss the applicability of this method for use in a modern aircraft to investigate the variation of g with height above sea-level.

(Oxford G.C.E. Scholarship level.)

21. Describe an accurate method of determining g, the gravitational acceleration. If the earth were a sphere of uniform density rotating with constant angular velocity, how would g vary from point to point on its surface? What variations are found in practice?

(Radius of earth = 6300 km.)

(Cambridge Univ. Schol., King's College Group.)

22. Describe an accurate method of measuring the acceleration due to gravity. Why is this measurement difficult to make at sea?

A spherical weight of 21 kg. is suspended by a spring balance. A second sphere weighing 160 kg. is brought beneath and the spring extends a further distance corresponding to the addition of 0·25 milligram when the centres are 30 cm. apart. Calculate the mass of the earth assuming the mean radius of the earth to be 6.10^8 cm.

(Oxford Univ. Schol.)

23. Outline the evidence for the universal truth of the inverse square law of attraction for gravitation.

A man can jump vertically 1·2 metres on the earth's surface. Calculate the maximum radius of a sphere of the same mean density as the earth from whose gravitational field he could escape by jumping.

(Radius of earth = $6{\cdot}37 \times 10^8$ cm.) (Oxford Univ. Schol.)

24. Discuss the contributions of Copernicus, Kepler and Newton to the study of gravitation and explain how the theory of universal gravitation was established.

Find the minimum velocity with which a projectile must be fired from the surface of the earth towards the moon so as to reach that planet. Ignore the effect of the motion of the earth and of the moon during the flight of the projectile.

(The moon describes an approximately circular orbit of mean radius

60·27 times the earth's radius which is $6{\cdot}37 \times 10^8$ cm. The mass of the earth is 81·53 times that of the moon.)

(Northern Univ. G.C.E. Schol. level.)

25. Give a short account of the evidence for Newton's law of gravitation.

Find the minimum velocities for a particle projected from a point on the surface of the earth in order that (*a*) it may become a satellite of the earth, (*b*) it may leave the earth altogether. What is the angle of projection required in each case? Radius of the earth = 6,500 km.

(Oxford Univ. Schol.)

26. A space ship is launched from the moon (mass m, radius r), with the object of reaching the earth (Mass M, distance from centre of moon to centre of earth = a). Find an expression for the least energy which must be expended per unit mass of the ship, in terms of the gravitational constant G.

On nearing the earth an engine is turned on to reduce the velocity of approach. What will be the apparent force of gravity within the ship (*a*) before the engine is turned on, (*b*) when the engine develops a thrust F on each unit mass of the ship? (Oxford Univ. Schol.)

CHAPTER 6

HYDROSTATICS

6.1 Introduction

'Hydrostatics' means the study of the effects which occur in fluids at rest. Many propositions can be demonstrated to be true merely by appealing to this fact, for if any effect can be shown to entail a motion of the fluid, then that effect cannot be present, since we are postulating throughout this chapter that the fluid must be at rest.

By a fluid is meant any substance which can flow, both liquids and gases. The results derived for one, however, do not always apply to the other, since liquids and gases differ in some important respects. Liquids are practically incompressible whilst gases can be compressed very easily, with a consequent change in density. Gases expand indefinitely to fill a container, but liquids, while they take up the shape of the vessel, are in general always bounded on one side by a free liquid surface. The effects of these differences are discussed as the chapter proceeds and must be carefully noted.

6.2 Pressure in a Fluid

The forces exerted by fluids are not usually concentrated at a point nor do they have a point of application, but they act uniformly over any surface placed in the fluid. Such a distributed force exerts a *Pressure* on the surface, pressure being defined as the component of a force acting normally to unit area, thus:

$$P = \frac{F}{A}$$

where F is the total force acting at right angles to a surface of area A and distributed evenly over it.

The dimensions of pressure are given by:

$$[P] = [MLT^{-2}/L^2]$$
$$[ML^{-1}T^{-2}].$$

The c.g.s. unit of pressure is the dyne.cm^{-2} and the f.p.s. unit is the poundal.ft^{-2} but the gravitational unit, lb-wt per square inch, is much more common. Other units of pressure are in use and are discussed on page 169.

Pressure is not a vector quantity, hence we cannot speak of pressure acting in any particular direction and certainly should not commit the common error of stating that the pressure in a fluid is equal in all directions.

It will be noticed that only the component of force normal to the surface was used in defining pressure; in fact, if the fluid is at rest, there can be no other component, for this would cause motion of the fluid.

By a similar argument, the same pressure must exist at all points in the same horizontal plane in a fluid at rest. If we take a thin horizontal slice, and assume that the pressure does vary along this slice, then motion of the fluid would ensue.

6.3 Variation of Fluid Pressure with Depth

In the vertical direction conditions are rather different, since the weight of the fluid must then be taken into account.

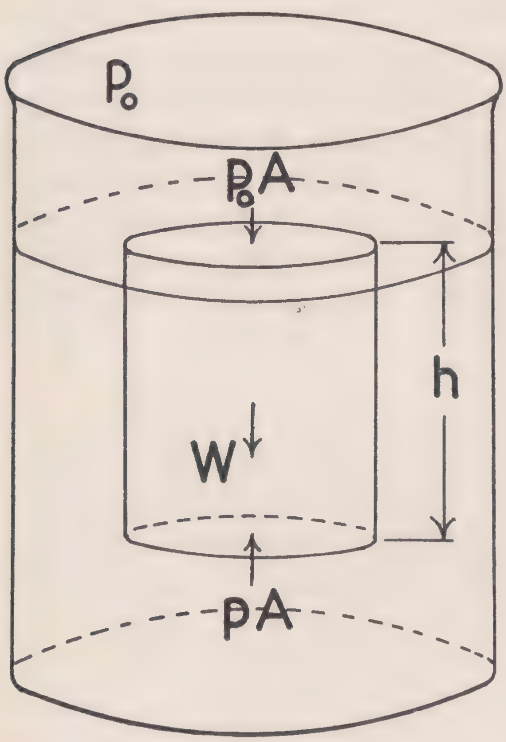


Fig. 6.1

Consider a liquid of density ϱ contained in a vessel, with the atmosphere exerting a pressure p_0 on the free surface, Fig. 6.1.

The forces acting on a cylinder of the liquid of height h and cross-sectional area A will be:

p_0A acting vertically downward due to the atmospheric pressure on the top surface.

W, the weight of the liquid within the cylinder acting downwards.

If p is the pressure in the liquid at depth h, there will be a force pA acting vertically upwards on the bottom of the cylinder.

For equilibrium (i.e. no motion of the fluid) in the vertical direction, all of these forces must cancel out, or

$$p_0A + W - pA = 0.$$

There will also be forces acting on the vertical walls of the cylinder, but these will be horizontal forces and so will not contribute to the vertical equilibrium.

$$\text{Now } W = Ah\varrho g,$$

$$\text{thus } p_0A + Ah\varrho g - pA = 0$$

$$\text{or } p = p_0 + g\varrho h \qquad (1)$$

which gives the variation in pressure below the surface of a liquid.

It will be noted that the pressure increases with depth, this is necessary to counteract the weight of the upper layers of liquid and so prevent vertical motion of the liquid.

6.4 Force on an Immersed Surface

The expression derived in the preceding section can be used to calculate the force exerted by a fluid on the walls of a containing vessel. The pressure in the fluid is constant in a horizontal plane, consequently the wall *ABCD* of the vessel (Fig. 6.2) can be divided into narrow horizontal strips of width δh (Fig. 6.3) and the pressure all along each one of these will be the same, although the pressure will increase from strip to strip as the depth in the fluid increases.

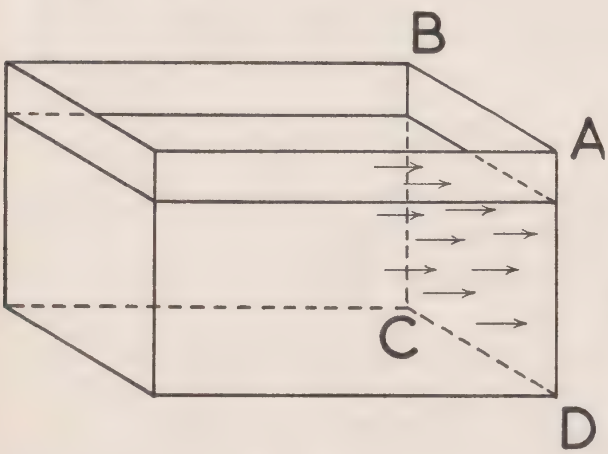


Fig. 6.2

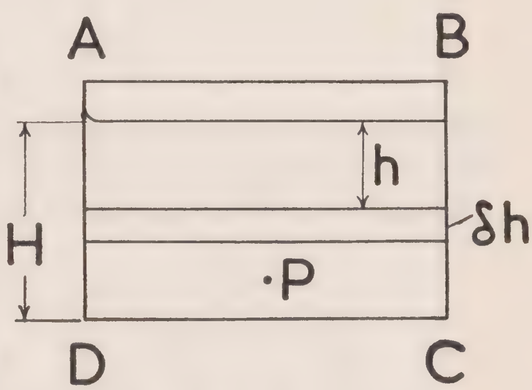


Fig. 6.3

For a typical strip at depth h in a liquid of density ϱ the pressure will be given by

$$p = g\varrho h$$

(the atmospheric pressure p_0 is neglected, since in general it will act on the outside of the wall as well and so cancel out).

The force acting on the strip is given by

$$\begin{aligned} \delta F &= \text{pressure} \times \text{area} \\ &= g\varrho h \,.\, a\delta h \end{aligned}$$

if the side AB is of length a. Consequently the total force on the whole wall is:

$$F = ag\varrho \int_0^H h dh,$$

where H is the total depth of the fluid.

$$\text{Thus } F = ag\varrho \frac{H^2}{2}.$$

It will be seen that aH is the area of the wall, and $g\varrho H/2$ is the pressure at the centre of gravity of the submerged portion of the wall, hence:

Force acting on a submerged surface = area $\times$ pressure at centre of gravity of surface . . . . . (2)

This result has been developed for a vertical rectangular surface, it is true, however, for a lamina of any shape and at any inclination.

It is also possible to calculate the point at which this force acts by taking moments about the bottom of the surface, then:

$$\text{Force on strip at depth } h = g\varrho h a \delta h.$$

Moment of this force about the bottom edge of surface

$$= g\varrho h a \delta h \,.\, (H - h),$$

$$\text{thus total moment of all forces} = a g \varrho \int_0^H h(H - h) dh$$

$$= a g \varrho \frac{H^3}{6}.$$

But the total force is equal to $ag\varrho H^2/2$ and if this acts at a distance d from the bottom it will have a moment $dag\varrho H^2/2$.

Equating this to the previous expression gives:

$$dag\varrho \frac{H^2}{2} = ag\varrho \frac{H^3}{6},$$

$$\text{thus } d = H/3 \text{ for a rectangle.}$$

The point at which the resultant force acts is called the *Centre of Pressure*, it is marked as the point P in Fig. 6.3. The position of the centre of pressure depends on the shape of the surface, and each one must be worked out on its merits using the method described above.

6.5 Pascal's Law

It will be seen from Equation (1) that if the pressure applied to the surface of the liquid is increased, then the pressure at every point in the liquid increases by the same amount; this fact was first stated by Pascal in his law: 'Pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and the walls of the containing vessel.'

6.6 Hydraulic Machinery

This transmission of pressure is of great use in many branches of engineering, for example many cars are fitted with hydraulically operated brakes. The brake pedal operates a piston producing pressure in an oil-filled cylinder and this pressure is transmitted by oil-filled tubes to cylinders fitted to the axles. The pressure then moves other pistons which push the brake shoes against the brake drums. Thus motion is transferred from the foot pedal to the brake shoes without a complicated system of brake rods and bell cranks. Moreover, it is *pressure* which is transmitted undiminished, hence by applying this pressure over large areas very large forces can be generated—much larger than the force used to produce the pressure in the first place.

6.7 Pressure Gauges

(a) Manometers

The pressure at the foot of a column of liquid can readily be calculated from the height of the column; this property enables the column to be used as a convenient form of pressure gauge. For example, if a U-tube is filled with liquid and connected to a vessel containing a gas at pressure p, as shown in Fig. 6.4, the liquid will stand higher in one limb than the other, and from the difference in height the pressure in the vessel can be calculated. A tube used in this fashion is called a *Manometer.*

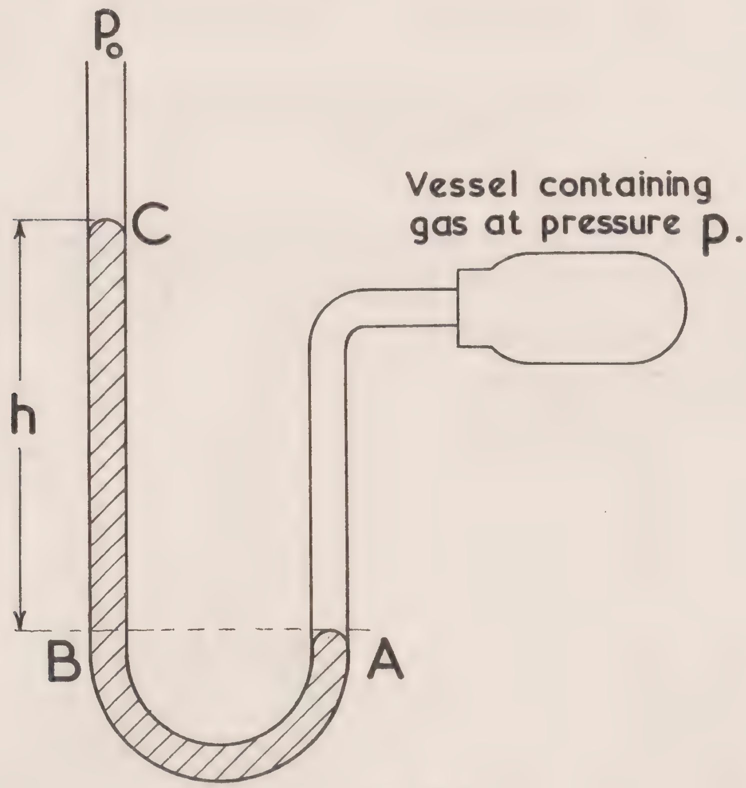


Fig. 6.4

The pressure on the surface at A is the pressure p in the vessel, and the pressure at B at the same horizontal level in the other limb will also be equal to p.

The pressure at B can also be written as $p_0 + g\varrho h$ if ϱ is the density of the liquid, for B is situated at a distance h below the surface C exposed to atmospheric pressure p_0, hence:

$$p = p_0 + g\varrho h,$$

and thus p can be found if p_0 and ϱ are known. This pressure is known as the Total or Absolute Pressure in the system.

More usually, we are interested only in the excess pressure over atmospheric pressure, i.e. in $p - p_0$ (often called the Gauge Pressure), which is given by:

$$p - p_0 = g\varrho h.$$

The difference in height of the two liquid columns is thus proportional to the excess pressure in the vessel.

This height is very often loosely referred to as being *equal* to the pressure in the vessel, thus the pressure in the gas mains might be said to be '5 inches of water'. A more accurate but more cumbersome statement of the case would be 'the excess pressure in the gas mains above atmospheric pressure is the same as the pressure found at the foot of a column of water 5 inches high'.

In this case the true excess pressure is given by:

$$p - p_0 = 32 \times 62{\cdot}5 \times \frac{5}{12} \text{ poundal.ft}^{-2}$$

$$= \frac{32 \times 62{\cdot}5 \times 5/12}{32 \times 144} \text{ lb-wt/sq in.}$$

$$\simeq \tfrac{1}{5} \text{ lb-wt/sq in.}$$

The modification necessary when pressures below atmospheric have to be measured is shown in Fig. 6.5. In this case $p = p_0 - \varrho gh$.

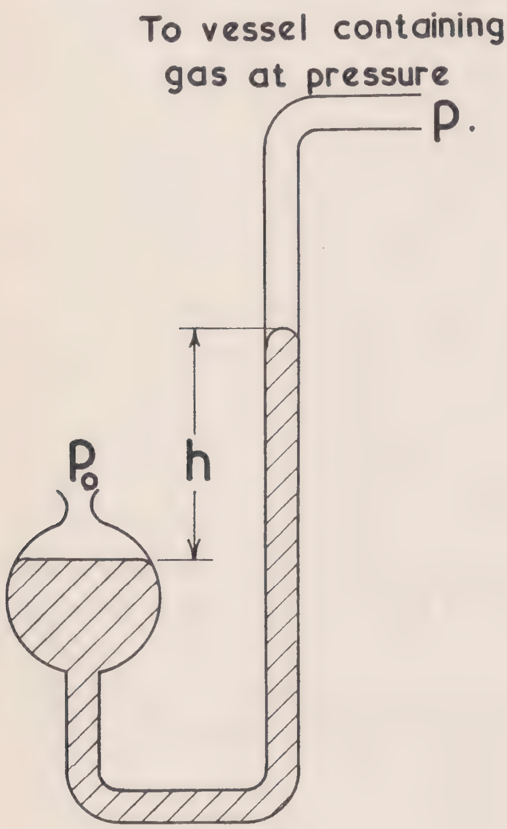


Fig. 6.5

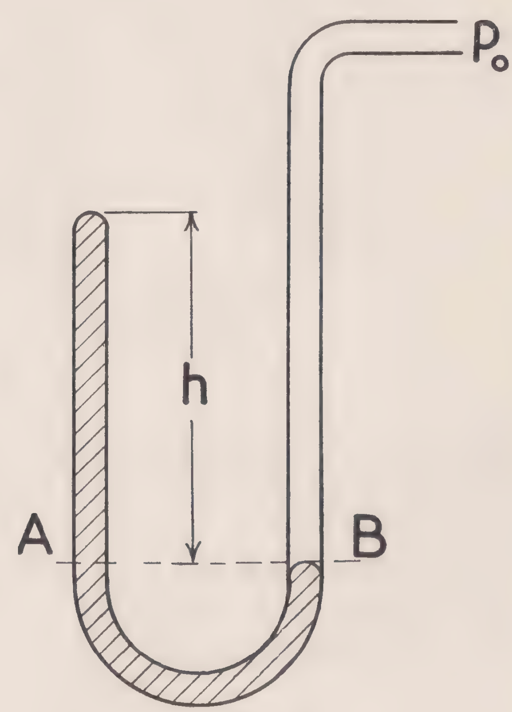


Fig. 6.6

Another variety of manometer which measures total pressure without reference to atmospheric pressure is shown in Fig. 6.6. This consists of a U-tube with one limb sealed off. Initially this tube is filled with liquid (usually mercury) so that the closed limb is completely filled as in Fig. 6.6—for this to be possible the closed limb must be fairly short.

If the pressure on the surface B is atmospheric, p_0, then the same pressure will exist at A in the other limb. The column of mercury above A can therefore exert no greater pressure than p_0, i.e.:

$$g\varrho h \leqslant p_0,$$

where ϱ is the density of mercury, or:

$$h_{max} = \frac{p_0}{g\varrho}.$$

The value of h_{max} for mercury is roughly 76 cm; thus the closed limb can be filled completely if it is shorter than this length.

If the open limb is now connected to a vessel containing air at a reduced pressure p, then the mercury will take up a position as in Fig. 6.7.

At the surface B′ the pressure is p, this pressure also appears at the same level A' in the other limb. But the pressure at A' is also $g\varrho h'$, since this point is at a distance h' below the free surface in the closed limb (note that above the mercury in this limb will be a vacuum, i.e. zero pressure), thus:

$$p = g\varrho h'.$$

As in a previous case, we very often speak of the height of the mercury column as being equal to the pressure—here the pressure would be described as 'h'' cm of mercury'. Note that 1 cm of mercury corresponds to a pressure of $1 \times 981 \times 13{\cdot}6$ dyne.cm^{-2}, i.e. 13,300 dyne.cm^{-2}.

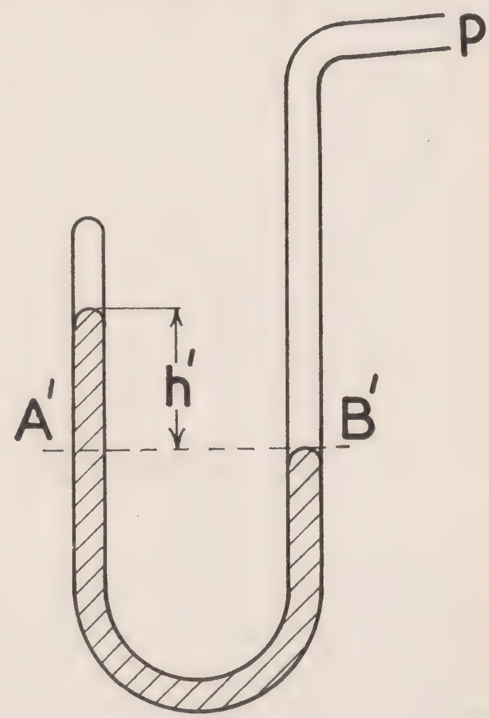


Fig. 6.7

The U-tube manometers are normally made of glass and are consequently rather fragile, moreover they must be used in the vertical position—these two factors limit the applications of the instruments considerably. In addition, the range of a manometer is limited—the U-tube manometer can be used for about 1 atmosphere excess pressure down to 1 cm of mercury absolute and the closed-limb manometer from 10 cm of mercury down to 1 mm absolute.

Other instruments have been designed to provide more robust pieces of apparatus for industrial use and to give an extended range. Some of these are described below, and those designed to extend the range down into the region of a 'high vacuum' appear in the next chapter.

(b) Bourdon Gauge

This is a robust instrument normally designed to measure high pressures. It consists of a piece of metal tube of oval cross-section

(Fig. 6.8), sealed at one end and bent into a spiral shape (Fig. 6.9).

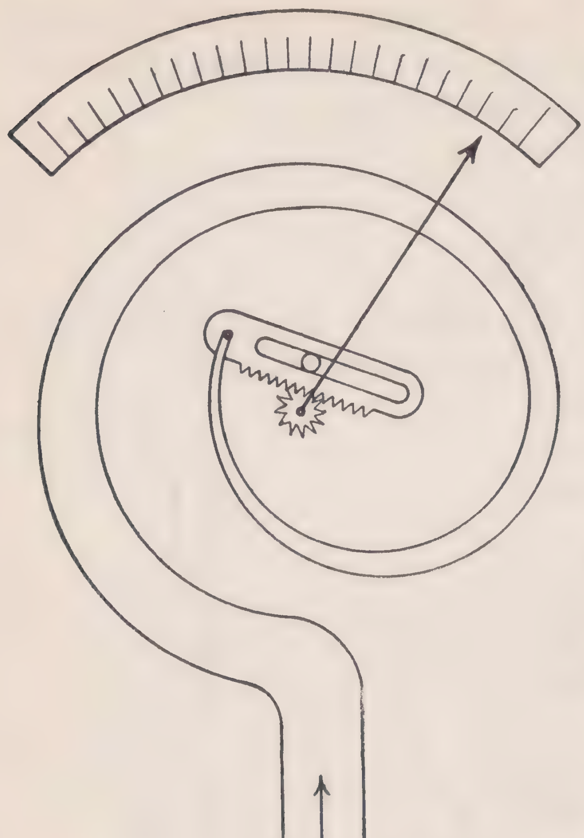


Fig. 6.9

If the pressure in this tube is increased it tends to uncurl, or if the pressure is reduced the tube curls up further; consequently if the end of the tube is fitted with a rack driving a pinion, a pointer can be made to move over a scale graduated in suitable pressure units, normally lb-wt per sq in. This gauge has to be calibrated by the manufacturer against a manometer.

(c) Capsule or Bellows Gauge

This type of gauge (usually used for measuring the lower pressures) consists of a cylindrical bellows made of very flexible metal. The bellows are evacuated and sealed off, consequently the atmospheric pressure outside squashes the bellows flat. The bellows are enclosed in a box (Fig. 6.10), and if the pressure in this box is reduced below atmospheric, the elasticity of the metal of the bellows makes it spring out again—the extent of the recovery depending on how far the pressure is reduced. The movement of the bellows is arranged to tilt a mirror and so move a spot of light over a scale which is previously calibrated against a manometer.

Fig. 6.8

Instead of the lamp and scale, this instrument is sometimes fitted with a mechanical linkage which enables the motion of the bellows to move a pointer over a scale.

6.8 The Barometer

When considering the closed U-tube manometer (Fig. 6.6) it was noticed that the atmospheric pressure could support a column of mercury only about 76 cm long in the closed limb; if the tube is any longer than this, the mercury drops as shown in Fig. 6.11, leaving a vacuum above the surface in the closed limb—actually this space is filled with

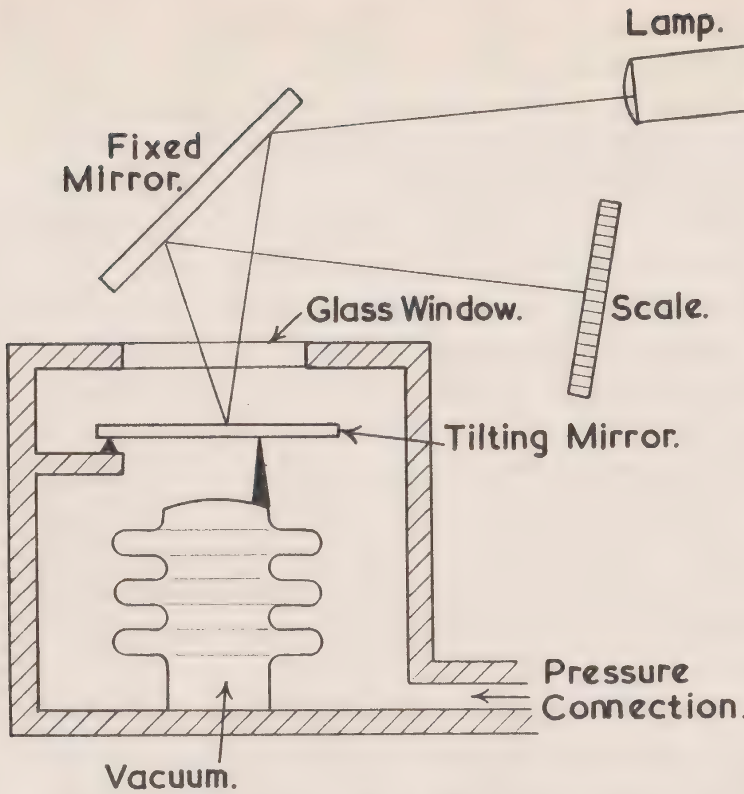


Fig. 6.10

mercury vapour (which at normal temperatures exerts a negligibly small pressure), and it is called a *Torricellian Vacuum.*

It was seen on page 165 that the pressure on the free surface in the open limb is given by

$$p_0 = g\varrho h,$$

hence, merely by making the closed limb rather longer than 76 cm, the instrument can be used to measure the atmospheric pressure—it is then called a *Barometer.*

Substituting numerical values in the equation above would give the atmospheric pressure in dyne.cm^{-2} but most commonly it is given merely as the height of the mercury column, i.e. the atmospheric pressure is said to be so many cm of mercury (but see Units of Pressure later in this chapter).

Usually a barometer is made by inverting a tube filled with mercury so that it stands in a dish (Fig. 6.12). For laboratory use it is mounted in a protective jacket as shown in Fig. 6.13. This type is known as a *Fortin's Barometer*; it carries a scale covering the usual range of barometric readings, and has a vernier moving over the scale. A mirror is fitted behind and parallel to the barometer tube; if both the edge of the vernier and its image in the mirror are made to coincide (at the same time) with the top of the mercury meniscus, then parallax is avoided in the setting of the vernier, and the barometric height can be read directly off the scale—usually to an accuracy of 1/20 mm.

The zero of the scale must of course coincide with the level of the mercury surface in the lower reservoir. Since the scale cannot be moved, the mercury in the lower reservoir is raised or lowered by a screw plunger which distorts the flexible bottom surface of the reservoir. The mercury surface is made to coincide with the tip of an ivory pointer, adjusted by the maker to be positioned accurately at the zero of the scale.

To obtain results of the greatest accuracy with this barometer, a temperature correction has to be applied for the expansion in length of the scale and for the change in density of the mercury; for details of this correction the reader is referred to text-books on Heat. Further, if readings of atmospheric pressure are to be intercompared between a number of stations (as when making a weather chart), a correction must be applied for the height of the barometer above sea-level since atmospheric pressure decreases with height.

Another form of barometer follows the same principle as the bellows manometer and is known as an *Aneroid Barometer* (Fig. 6.14). The flexure of the bellows under variations of atmospheric pressure is transmitted by a mechanical linkage designed to move a pointer over a scale, otherwise the action of the instrument is exactly the same.

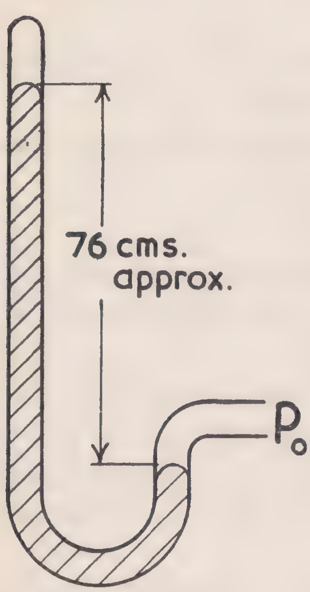


Fig. 6.11

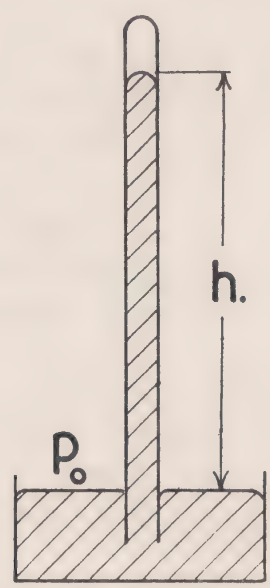


Fig. 6.12

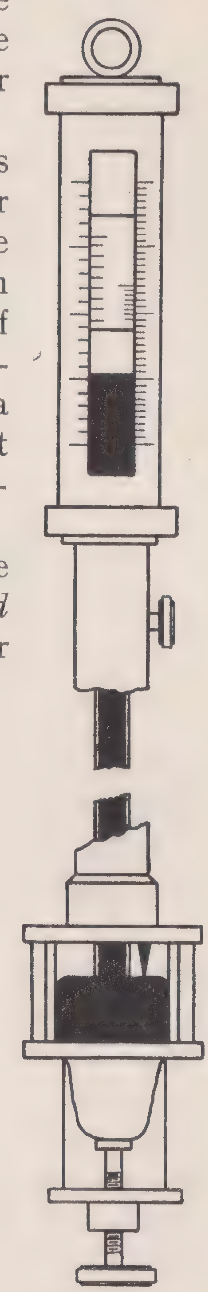


Fig. 6.13

The aneroid barometer has to be calibrated against a Fortin barometer in the first place.

The aneroid barometer can be made comparable in accuracy with the Fortin, moreover, once it is accurately calibrated it will give correct

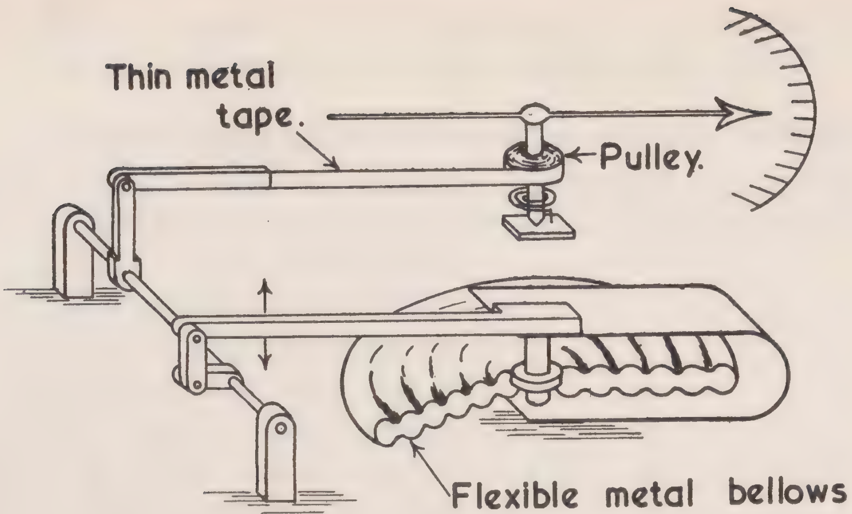


Fig. 6.14

readings of pressure wherever it may be situated; the height of the mercury column in a Fortin barometer, however, is given by:

$$h = \frac{p_0}{g\varrho}$$

and thus, *for the same atmospheric pressure*, the height of the mercury barometer will vary inversely with g.

If both varieties of barometer are available at the same location, p_0 can be read (in dyne.cm^{-2}) from the aneroid barometer, and h (in cm) from the mercury barometer, and hence g for the locality can be calculated from:

$$g = \frac{p_0}{\varrho h}.$$

This can be made to give quite accurate values for g and is used to extend gravity surveys to 'difficult' places (e.g. at sea) where a pendulum cannot be used.

6.9 Units of Pressure

It has already been seen that pressure may be expressed in dyne.cm^{-2}, lb-wt per sq in., or loosely in cm of mercury. Normal atmospheric pressure is approximately 15 lb-wt/sq in. Standard atmospheric pressure is taken as 76 cm of mercury at 0° C. and using a standard value for g, this converts to 14·7 lb-wt/sq in.; this is called '1 *atmosphere*'. Thus a pressure of 5 atmospheres is 73·5 lb-wt/sq in. This unit is obviously useful when dealing with high pressures.

If 76 cm of mercury is converted to pressure in the c.g.s. system it becomes $1{\cdot}013 \times 10^6$ dyne.cm^{-2}, or 1 atmosphere $= 1{\cdot}013 \times 10^6$ dyne.cm^{-2}.

A pressure of 10^6 dyne.cm^{-2} is also called '1 *bar*', subdivided into

1000 *millibars,* thus 1 atmosphere = 1013 millibars. The millibar is now used to give atmospheric pressure on all weather charts.

6.10 Variation of Atmospheric Pressure with Height

The existence of atmospheric pressure has been used in several preceding sections without advancing any explanation for it. The surface of the Earth is covered with a layer of fluid (the air) many miles deep; we are thus situated below the surface of this fluid and consequently feel a pressure due to the weight of the column of air above us. Unfortunately this pressure cannot be equated to $g\varrho h$ as in the case of a liquid, because air is very compressible and its density is by no means a constant quantity. Instead we can proceed as follows.

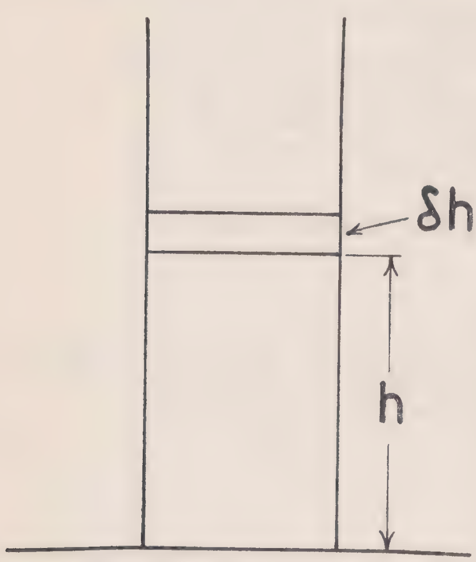


Fig. 6.15

Consider a column of air (Fig. 6.15) above the surface of the Earth. At distance h above the surface, take a slab of this column of thickness δh and let the air density at this point be ϱ. Then the increment of pressure in going from height h to $h + \delta h$ is given by:

$$\delta p = -g\varrho\delta h$$

(negative because the pressure *decreases* as h increases).

The density of the air can be calculated in terms of its pressure, using the Gas Laws (see Chapter 7) and is given by:

$$\varrho = \frac{pM}{RT},$$

where M is the molecular weight of the gas, R the gas constant for a gram-molecule and T the absolute temperature.

$$\text{Thus } \delta p = -\frac{gpM\delta h}{RT}$$

$$\text{or } \frac{\delta p}{p} = -\frac{gM}{RT}\delta h \quad . \quad . \quad . \quad . \quad (3)$$

This is a differential equation relating change of pressure to height. The actual equation linking these two variables can be found by integration only if we know how T varies with h. The variation of temperature with height is very irregular, but if we replace T by its average value $\overline{T}$ up to height h, we can solve the equation and hence get a rough approximation to the actual fall of pressure with height.

Making this approximation and integrating Equation (3) gives:

$$\log_e p = -\frac{gMh}{R\overline{T}} + \text{const},$$

and if $p = p_0$ when $h = 0$, i.e. at Mean Sea Level, then:

$$\text{const} = \log_e p_0,$$

$$\text{thus } \log_e p = -\frac{gMh}{R\overline{T}} + \log_e p_0.$$

This leads to:

$$\log_e \frac{p}{p_0} = -\frac{gMh}{R\overline{T}}$$

$$\text{or } p = p_0 \exp\left(-\frac{gMh}{R\overline{T}}\right) \quad . \quad . \quad . \quad (4)$$

[The symbol $\exp(x)$ is merely a more convenient way of printing e^x.] Equation (4) is known as the *Barometric Equation* for an isothermal atmosphere and can be used to correct barometric readings taken at a known height to the corresponding Mean Sea Level value, provided only that the observing station is not so high that T decreases considerably.

The factor $(gM/R\overline{T})$ has the value $1{\cdot}25 \times 10^{-5}$ for air at normal temperatures, so that

$$p = p_0 \exp(-1{\cdot}25h \times 10^{-5}) \quad . \quad . \quad . \quad (4a)$$

In this equation the height is to be measured in centimetres.

6.11 Archimedes' Principle

A solid immersed in a fluid experiences a pressure all over its surface, this pressure increases with depth in the fluid, and thus the forces exerted by the fluid pressure will be greater at the bottom of the solid than at the top. The solid therefore experiences an upthrust.

This fact is stated in *Archimedes' Principle*: When a solid body is wholly or partially immersed in a fluid, then it experiences an upthrust equal to the weight of the displaced fluid. This upthrust acts vertically through the centre of gravity of the displaced fluid.

This law suffers a great deal of mis-statement by students; the most common errors are: the upthrust is equal to the 'amount' or 'volume' of the displaced fluid, and the upthrust is equal to the weight of the displaced *water*.

The upthrust experienced by an immersed body opposes the gravitational pull on it and so the apparent weight of a body when submerged is less than its true weight.

6.12 Flotation

If the upthrust on a totally immersed body is greater than its true weight, the body will move upwards towards the surface. This will

happen if the fluid is denser than the solid, when the apparent weight of a totally immersed body becomes negative. As the body breaks surface, the volume of the submerged part decreases, consequently the upthrust also decreases until a point is reached where it exactly balances the weight of the solid and the body is then floating on the surface.

If in this position there is a volume V' of the solid immersed, the upthrust is given by $V'\sigma g$ where σ is the density of the liquid. For equilibrium, the upthrust acting on the body must be equal to its weight, thus if the total volume of the solid is V and its density is ϱ, we have:

$$V\varrho g = V'\sigma g,$$

$$\text{giving } \frac{V}{V'} = \frac{\sigma}{\varrho}.$$

A floating body takes up a position so that it displaces its *own* weight of the fluid on which it floats.

6.13 Stability of Floating Bodies

A floating body is normally under the action of two forces: its weight acting vertically downwards through its centre of gravity, and the upthrust acting vertically through the centre of flotation. If the body is to be in equilibrium, these two forces must be equal, opposite in direction and act in the same straight line, as shown in Fig. 6.16.

The equilibrium position may or may not be stable (see page 66)—as a test for stability we displace the body through a small angle and see whether or not it returns to its equilibrium position.

If the body is displaced about the point O as in Fig. 6.17 the centre of gravity G (which is a fixed point in the body) moves over to the right of O. The submerged part of the body is now wedge shaped and the centre of flotation (centre of gravity of the displaced fluid), moves to a point such as F. This may or may not be to the right of O, depending on the geometry of the figure.

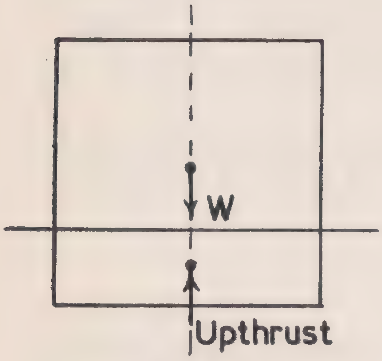


Fig. 6.16

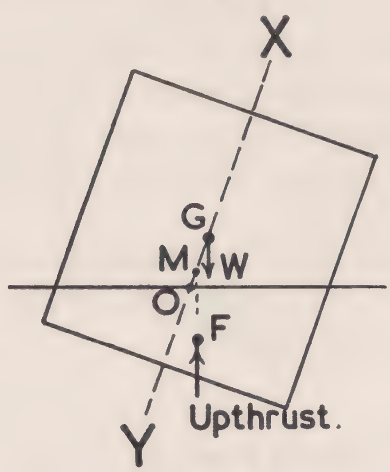


Fig. 6.17

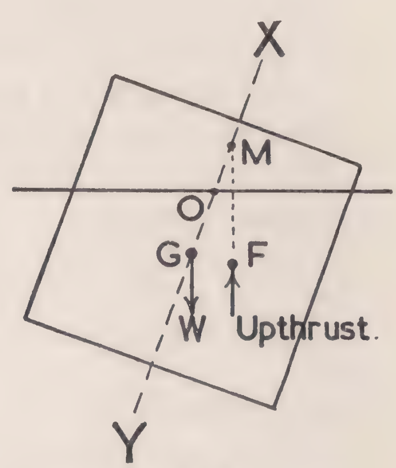


Fig. 6.18

The weight of the body acts through G and the upthrust through F; these forces will not in general act in the same straight line and will consequently form a couple. In Fig. 6.17 the couple is clockwise and will tend to topple the body over—the equilibrium is thus unstable; if, however, the body is denser and floats lower in the fluid as in Fig. 6.18, then the couple will be anticlockwise and tends to right the body, therefore the equilibrium is now stable.

As a criterion of whether the equilibrium is stable or not, tilt the body into the displaced position and notice where the vertical line through the centre of flotation cuts the line XY (drawn vertically on the body through the centroid when in the rest position). If this point of intersection, M, is below the centre of gravity as in Fig. 6.17, then the weight and upthrust must form a toppling couple and the equilibrium is unstable. If M is above G (Fig. 6.18), they form a righting couple and the position is stable. M is called the *Metacentre* of the body.

In the rare case when the metacentre and the centre of gravity of the body coincide, the equilibrium is neutral and the body, if displaced, will stay in the displaced position.

6.14 Air Buoyancy

The loss of weight which occurs when a body is immersed in a fluid occurs even when the fluid is the air of the atmosphere; it is the upthrust produced by displaced air which is used to lift balloons and airships. The effect is not large owing to the small density of air, but can be enhanced by using very large balloons which displace a lot of air.

(a) Open Balloons

The balloons used by the early pioneers had an envelope which could be supported in a spherical shape by a very small excess internal pressure; the gas filling was confined to the envelope by means of a valve set to maintain a constant *excess* pressure, and so permitted the gas to escape as the balloon ascended and the atmospheric pressure became less. These are called *Open Balloons.*

In practice the excess pressure inside the balloon is so small that it can be neglected and the internal and external pressures treated as equal.

If the envelope has a mass m and volume V, and is filled with gas of density ϱ_1, while the density of air is ϱ_2, then the total weight of the balloon is given by

$$\begin{aligned} W &= \text{weight of envelope} + \text{weight of contained gas} \\ &= (m + V\varrho_1)g \qquad (5) \end{aligned}$$

But upthrust due to air = weight of displaced air

$$= V\varrho_2 g \qquad (6)$$

Now the lifting capacity of the balloon is given by:

$$\text{Lift} = \text{upthrust} - \text{weight of balloon} \qquad (7)$$
$$= \{V(\varrho_2 - \varrho_1) - m\}g$$
$$= \left\{V\varrho_2\left(1 - \frac{\varrho_1}{\varrho_2}\right) - m\right\}g \qquad (8)$$

If both of the gases obey Boyle's Law (see Chapter 7), they will expand to the same extent as the pressure decreases and consequently their densities will be reduced at the same rate. The ratio ϱ_1/ϱ_2 will therefore stay constant, so also will the factor $(1 - \varrho_1/\varrho_2)$. However ϱ_2 decreases with height, consequently the balloon will reach a height where the term $V\varrho_2\,(1 - \varrho_1/\varrho_2)$ is equal to m,

$$\text{i.e. } V\varrho_2\,(1 - \varrho_1/\varrho_2) - m = 0.$$

Consequently the lift of the balloon is zero and it has reached its ceiling. If this occurs for a value of $\varrho_2 = \varrho_{\text{ceiling}}$, then:

$$\varrho_{\text{ceiling}} = \frac{m}{V(1 - \varrho_1/\varrho_2)} \qquad (9)$$

and assuming an isothermal atmosphere, this value of density can be converted into a height by the use of the Barometric Equation (4a).

It should be emphasised that the presence of a gas inside the envelope is merely to hold it in its spherical form, the filling gas contributes nothing to the lift of the balloon, in fact it weighs the balloon down—a much better filling would be a vacuum if a suitable envelope could be devised. *The entire lift comes from the displaced air.*

(b) Closed Balloons

Nowadays balloons are used to obtain meteorological information from the stratosphere and also for cosmic-ray research at great heights.

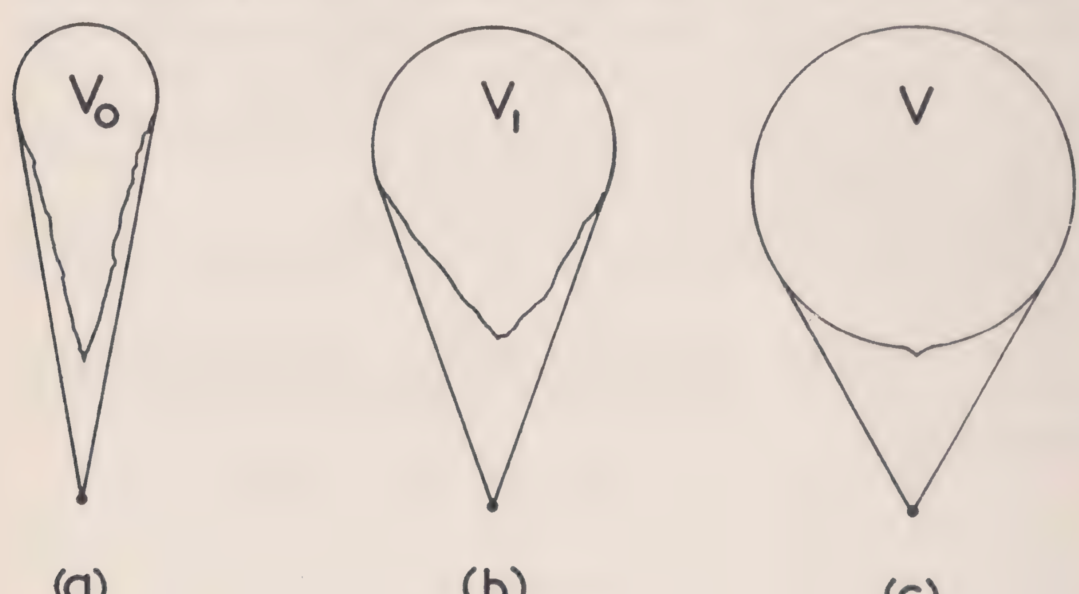


Fig. 6.19

At ground-level these balloons appear to be partially deflated (Fig. 6.19 (*a*)). These are *Closed Balloons* and are filled with a fixed mass M of gas.

Let the gas occupy a volume V_0 at ground-level where the air pressure is p_0 (the envelope is completely yielding, so that internal and external pressures are the same); then if the density of the air at ground-level is ϱ_0, the upthrust, which is equal to the weight of air displaced, is given by $V_0\varrho_0 g$. The weight of the balloon is $(M + m)g$, where m is the mass of the envelope, and thus the lifting power is given by:

$$\text{lift} = \Big(V_0\varrho_0 - (M + m)\Big)g.$$

If the balloon now ascends to a region where the atmospheric pressure falls to p_1, the gas in the balloon expands to a new volume V_1 (Fig. 6.19 (*b*)) given by:

$$p_0V_0 = p_1V_1 \text{ (see Chapter 7)},$$

and the density of the air changes to a value ϱ_1, where:

$$\frac{p_1}{p_0} = \frac{\varrho_1}{\varrho_0} \text{ (see Chapter 7)}.$$

The upthrust is now equal to $V_1\varrho_1 g$ and substituting from the equations above, this becomes:

$$\begin{aligned} \text{upthrust} &= \frac{p_0V_0}{p_1} \cdot \varrho_1 g \\ &= \frac{p_0V_0}{p_1} \cdot \frac{p_1\varrho_0}{p_0} \cdot g \\ &= V_0\varrho_0 g \text{ as before,} \end{aligned}$$

and, again, the lifting power is given by:

$$\text{lift} = \{V_0\varrho_0 - (M + m)\}g \qquad . \qquad . \quad (10)$$

The balloon thus produces a *constant lift*, whatever its height, provided only that the envelope can expand to equalise the internal and external pressure.

If the gas in the balloon expands so much that the envelope is fully distended and takes up a fixed volume V (Fig. 6.19 (*c*)), then the assumption of equality of pressure no longer holds. If the air density at this height is ϱ_2, the lifting capacity of the balloon is given by:

$$\text{lift} = \{V\varrho_2 - (M + m)\}g \qquad . \qquad . \quad (11)$$

All of these quantities are constant except ϱ_2 which decreases with height, and thus a ceiling is reached when

$$\begin{aligned} V\varrho_{\text{ceiling}} &= (M + m) \\ \text{or } \varrho_{\text{ceiling}} &= \frac{M + m}{V} \qquad . \quad . \quad . \quad . \quad . \quad (12) \end{aligned}$$

To summarise, then, the closed balloon produces a constant lift, given by Equation (10), so long as the envelope is relaxed; when the envelope starts to stretch, however, the lift gradually reduces to the value given by Equation (11), when the envelope is fully distended; the balloon thereafter continues to rise until it reaches its ceiling, given by Equation (12).

The ceiling of the closed balloon is slightly lower than that of an open balloon of the same volume, since the closed balloon is bound to retain a larger mass of gas; it is, however, much more economical of gas, since it is filled with just the correct amount of gas to get it up to its ceiling, while the open balloon is filled completely at ground-level and has to release gas throughout the whole of its ascent. The magnitude of this saving can be assessed when it is realised that a closed balloon used for stratosphere ascents needs several hundredweights of hydrogen for its filling.

Example 1. *A partly inflated closed balloon contains* 200 *kg of hydrogen. Find the greatest load that it can lift if the envelope weighs* 500 *kg.* (*Density of air at sea-level* = 1·29 *gm per litre, density of Hydrogen* = 0·09 *gm per litre.*)

The volume of the balloon at sea-level is equal to $\dfrac{200 \times 1000}{0 \cdot 09}$ litres

$$\text{or } V_0 = \frac{200 \times 1000 \times 1000}{0 \cdot 09} \text{ cm}^3,$$

assuming that 1 litre = 1000 cm³,

thus $V_0 = 2 \cdot 2 \times 10^9$ cm³.

Substituting numerical values in Equation (10), gives:

$$\text{lift} = [2 \cdot 2 \times 10^9 \times 1 \cdot 29 \times 10^{-3} - (2{,}000{,}000 + 500{,}000)]\, g \text{ dynes.}$$

(Notice that the density of air must be converted to the fundamental unit gm.cm^{-3}, not gm per *litre*).

$$\begin{aligned}\text{Thus lift} &= (2 \cdot 84 \times 10^6 - 2 \cdot 5 \times 10^6)\text{gm-wt}\\ &= 0 \cdot 34 \times 10^3 \text{ kg-wt}\\ &= 340 \text{ kg-wt.}\end{aligned}$$

(c) Air Buoyancy Correction when Weighing

The density of air is about 1·3 gm per litre, thus a body of volume 100 cm³, a calorimeter containing water, for example, experiences an upthrust or loss of weight of about 0·13 gm-wt. If this calorimeter is counterpoised on a balance against very small brass weights which do not displace as much air as the water, then a considerable error is introduced into the weighing, but an allowance can be made for the error as follows.

A body of mass M and density ϱ is to be weighed and is balanced by a 'weight' of mass m and density σ. Let the air density be ϱ_0.

$$\begin{aligned}\text{Then true weight of body} &= Mg,\\ \text{but volume of body} &= M/\varrho\\ &= \text{volume of air displaced by body,}\end{aligned}$$

$$\text{thus weight of air displaced} = \frac{M}{\varrho}\varrho_0 g,$$

$$\begin{aligned}\text{and apparent weight of body} &= Mg - \frac{M\varrho_0 g}{\varrho}\\ &= Mg\left(1 - \frac{\varrho_0}{\varrho}\right).\end{aligned}$$

$$\begin{aligned}\text{Similarly true weight of 'weight'} &= mg,\\ \text{but volume of 'weight'} &= m/\sigma\\ &= \text{volume of air displaced by weight,}\\ \text{thus weight of air displaced} &= \frac{m}{\sigma}\varrho_0 g,\\ \text{and apparent weight of 'weight'} &= mg\left(1 - \frac{\varrho_0}{\sigma}\right).\end{aligned}$$

It is these apparent weights which are balanced by the balance, hence:

$$Mg\left(1 - \frac{\varrho_0}{\varrho}\right) = mg\left(1 - \frac{\varrho_0}{\sigma}\right)$$

$$\begin{aligned}\text{or } M &= m\left(\frac{1 - \dfrac{\varrho_0}{\sigma}}{1 - \dfrac{\varrho_0}{\sigma}}\right)\\ &= m\left(1 - \frac{\varrho_0}{\sigma}\right)\left(1 - \frac{\varrho_0}{\varrho}\right)^{-1}.\end{aligned}$$

Now ϱ_0/σ and ϱ_0/ϱ are both small terms; thus expanding the last bracket by the binomial theorem and ignoring terms in $(\varrho_0/\varrho)^2$ and higher powers gives:

$$M = m\left(1 - \frac{\varrho_0}{\sigma}\right)\left(1 + \frac{\varrho_0}{\varrho}\right).$$

Multiplying these brackets together and ignoring products of small terms gives:

$$\begin{aligned}M &= m\left(1 - \frac{\varrho_0}{\sigma} + \frac{\varrho_0}{\varrho}\right)\\ &= m\left\{1 - \varrho_0\left(\frac{1}{\sigma} - \frac{1}{\varrho}\right)\right\} \qquad (13)\end{aligned}$$

The factor $1 - \varrho_0\left(\dfrac{1}{\sigma} - \dfrac{1}{\varrho}\right)$ is a correction term by which the mass read off from the weights of the balance pan must be multiplied in order to get the true mass of the body which is being weighed.

This correction is of surprisingly large size, but is very often neglected; most of us will recall laboriously weighing a calorimeter of water to the nearest milligram and then neglecting to apply the buoyancy correction which generally invalidates all figures from 0·1 gm onwards.

SUMMARY OF NEW UNITS INTRODUCED IN THIS CHAPTER

Quantity	*c.g.s. unit*	*f.p.s. unit*	*M.K.S. unit*	*Gravitational and other units*
Pressure	dyne.cm^{-2}	poundal.ft^{-2}	newton.metre^{-2}	gm-wt per sq cm kg-wt per sq cm lb-wt per sq in. cm of mercury bars and millibars

EXERCISES 6

1. Define *pressure*, and explain what is meant by the *pressure at a point* in a fluid.

   Describe an accurate form of mercury barometer, and explain carefully the physical principles underlying the use of such an instrument for the measurement of atmospheric pressure.

   A mercury barometer is taken down a mine-shaft in an open lift-cage. The deceleration of the cage when it has descended a distance of 500 metres is 100 cm.sec.$^{-2}$ Find the reading of the barometer at this instant, given that the steady reading at the top of the shaft is 75·0 cm.

   (Take the density of mercury as 13·6 gm.cm.$^{-3}$ and the density of the air as 0·0013 gm.cm^{-3}.) (Oxford H.S.C.)

2. Describe an accurate form of barometer and illustrate your description with a sketch. Indicate how a vernier may be used to obtain an accurate reading. Explain the corrections which have to be applied to the reading observed to make it suitable for meteorological purposes. (Cambridge H.S.C.)

3. A fountain is supplied from a water main in which the absolute pressure is two atmospheres. Estimate the height of the water jet. (Cambridge Univ. Schol., King's College Group.)

4. Explain what is meant by the pressure at a point in a fluid.

   A manometer consists of two narrow vertical limbs of length l, distant d apart, connected at their lower ends by a horizontal tube. Both limbs are open to the atmosphere, and the manometer contains mercury to a depth $\frac{1}{2}l$ in each limb. The manometer is rotated about the vertical axis of one limb with angular velocity ω ($<\sqrt{2gl/d^2}$). Find the difference in height of the two mercury surfaces.

   Discuss what happens when $\omega > \sqrt{2gl/d^2}$. (Oxford Univ. Schol.)

5. State Archimedes' principle, and describe how you would verify it experimentally. A deep vessel is filled, half with water and half with benzene (sp.gr. 0·88), which is immiscible with water. A thin cylinder, of length 12 cm. and sp.gr. 0·92, is placed in the vessel and constrained to move with its axis vertical. At what position does the cylinder come to rest? If the cylinder be displaced vertically from this position and released, what will be the period of the subsequent oscillations? The effects of viscosity may be neglected.

(Cambridge Univ. Schol., King's College Group.)

6. What is Archimedes' Principle?

A diving bell consists of a thin-walled hollow right circular cylinder 10 feet in diameter, with one open end. It is lowered into water with its axis vertical and the open end downwards. When the open end is 42 feet below the surface the bell is half full of water and does not tend to rise or sink. Find the weight and length of the bell. Discuss the effect of a small vertical displacement from this position.

(Height of water barometer 33 feet. 1 cu. ft. of water weighs 62·5 lb.) (Oxford Univ. Schol. (Subsid.).)

7. A uniform solid cone is floating in water with its axis vertical and its vertex immersed. A weight of 36 lb. placed on the base causes it to sink until $\frac{4}{5}$ of the axis is immersed and a weight of $62\frac{1}{2}$ lb. causes $\frac{5}{6}$ of the axis to be immersed. Find the specific gravity of the material of the cone and the weight required to sink it completely.

(London Univ. Inter. B.Sc.)

8. Describe a hydrometer suitable for the determination of the specific gravity of spirits. How would you test the graduation of a hydrometer?

A mixture of alcohol (sp.gr. 0·84) and water has a specific gravity of 0·90. What is the proportion of the constituents (*a*) by weight, (*b*) by volume (neglecting any change of volume which occurs on mixing?) (Cambridge H.S.C.)

9. Explain the principle of the common hydrometer.

Design a hydrometer, of total mass 50 gm., which is to have a scale 15 cm. long, graduated for the measurement of specific gravities between 1·00 and 0·85.

This hydrometer is floating in water, and is pushed down until the water-level reaches the 0·85 mark. Find the force required to keep it there, and the work done in moving it down.

(Oxford G.C.E. Schol. level.)

10. Define density and specific gravity. Describe how you would measure the specific gravity of a liquid. Examine the sources of error in the method you describe.

A lump of metal is suspended from the beam of a balance and weighs 47·93 gm. It is then immersed in water and it is found to be necessary to add 5·70 gm. to one balance pan to keep the beam balanced. Calculate the specific gravity of the metal.

(Cambridge Univ. Schol., Girton and Newnham Colleges.)

11. A vessel contains 1,200 c.c. of saturated brine, of sp.gr. 1·2, and above this a layer of 400 c.c. of water, which has been introduced in such a manner as to avoid appreciable mixing. A hollow glass bulb of mass 21·5 gm. and volume 20·0 c.c. is in equilibrium in the vessel and is entirely surrounded by liquid; find the volume of the bulb immersed in each liquid. The whole is now thoroughly stirred; find the volume of the bulb immersed in the resulting solution. (Assume that there is no volume change when the liquids are mixed.)

Oxford H.S.C. (Part).

12. (*a*) Explain how the weight of a motor-car is supported by the tyres.

(*b*) A test tube of radius a is weighted with lead shot so that it floats upright in water, the total mass being m. Find an expression for the period of small oscillations which ensue when the test-tube is slightly depressed from its equilibrium position. Discuss briefly the effects of surface tension and viscosity on the motion.

(Oxford Univ. Schol.)

13. A Nicholson's hydrometer floats in water with its mark in the surface when the load in the upper pan is 10 gm., and when a further 0·1 gm. is added the hydrometer floats with the mark 2 cm. below the surface. The volume of the instrument up to the mark is 50 c.c. Calculate the period of small vertical oscillations of the hydrometer when displaced from the normal equilibrium position (*a*) in water, (*b*) in a liquid of density 0·9 gm. per c.c., (*c*) in water with a 10-gm. load of material of density 8 gm. per c.c. in the lower pan. Derive any formula you employ. (Cambridge Univ. Schol., King's College Group.)

14. Show that if the temperature of the atmosphere is uniform, its density ρ at a height h above the earth's surface is given by the relationship $\rho = \rho_0 e^{\frac{-h}{H}}$, where ρ_0 is the density at sea-level and H is a constant.

A meteorological balloon is spherical in shape and is constructed of inextensible fabric of mass 0·1 gm.cm.$^{-2}$. It is required to carry recording apparatus of negligible weight to a height of 8 km. What must be its minimum radius?

(1 gm. molecule of hydrogen at sea-level occupies a volume of 22·4 litres. Gm. molecular weight of air $= 29$. $H = 8$ km.)

(Manchester Univ. Schol.)

15. Discuss the equilibrium of floating bodies, with special reference to ships. Explain what is meant by the metacentre, and prove the formula Ak^2/v for its height above the C.G., where the symbols have their usual significance. (Oxford G.C.E. Schol. level.)

16. A rectangular aperture in a vertical lock gate is closed by a flat plate 5 feet wide and 4 feet deep, the longer edges being horizontal. Find the resultant thrust of the water on the plate in lb.wt. and its line of action,

(i) when the water-level on one side of the gate is 1 foot below the top edge of the plate and on the other side just reaches the top edge of the plate;

(ii) when the difference in the water-levels on opposite sides of the gate is 3 feet and both levels are above the top edge of the plate.

(London Univ. Inter. B.Sc.)

17. A vertical canal gate is 10 ft. broad. On one side the water is 12 ft. deep and on the other it is 6 ft. deep, measured from the bottom of the gate. Calculate the magnitude and position of the resultant hydrostatic force on the gate. (Take 1 cu. ft. of water to weigh 62·5 lb.)

(London Univ. Inter. B.Sc.)

CHAPTER 7

PROPERTIES OF GASES

7.1 Introduction

The study of the laws governing the behaviour of gases when subjected to variations of pressure and temperature can rightly be included either in a textbook of General Physics or of Heat. In this volume little more is done than to state the gas laws and then to use them to explain the operation of various physical systems such as air pumps and vacuum gauges. For a theoretical treatment of the gas laws, the student is referred to the companion volume, *Heat,* by A. J. Woodall, Ph.D., published by English Universities Press, Ltd.

At the end of this chapter it is shown how the behaviour of a gas can be explained by applying the ideas of mechanics to the motion of the molecules which constitute the gas.

7.2 Permanent Gases and Vapours

The first systematic investigations of the behaviour of gases were made by Boyle in 1650–60; his experiments were facilitated by a 'Pneumatical Engine', or air pump, invented by von Guericke in 1654. Boyle discovered that air had weight and also that if the pressure of a given mass of air was changed, then its volume changed in inverse proportion to the pressure; this fact has now become known as *Boyle's Law.*

Subsequently it was discovered that some gases when compressed behaved in this fashion, while others turned to a liquid; these were called *permanent gases* and *vapours* respectively. The name 'permanent gas' is, however, an unfortunate one, since a later discovery has shown that all gases will liquefy on compression if first cooled below a certain *critical temperature.* A permanent gas is thus a gas above its critical temperature, while a vapour is a gas below its critical temperature.

7.3 Vapour Pressure

When the space occupied by a vapour is reduced, the pressure in the vapour first of all increases until it reaches a value called the *saturation vapour pressure*; thereafter any decrease in volume causes the vapour to liquefy while the vapour pressure remains constant at its saturation value. The magnitude of the saturation vapour pressure depends on the nature of the vapour and the temperature. Reasons for this are advanced in Chapter 9.

7.4 The Gas Laws

(a) Dalton's Law of Partial Pressures

Nearly 150 years after Boyle's work on gases, Dalton and Regnault individually performed experiments to measure the pressure exerted by a vapour alone and by a vapour mixed with a gas; they found that the pressure exerted by the vapour was the same in each case, and the results are stated in Dalton's *Law of Partial Pressures*:

When a mixture of gases or vapours, having no chemical interaction, are present together in a given space at a given temperature, the pressure exerted by each constituent is the same as if it alone filled the whole of the space. The total pressure is the sum of the partial pressures due to the constituents.

(b) Boyle's Law

Boyle enunciated the following law summarising the results of his experiments:

At a constant temperature, the product of the volume of a given mass of gas and the pressure to which it is subjected remains constant as the pressure and volume are varied.

$$\text{Thus } pv = \text{constant} \qquad . \quad . \quad . \quad (1)$$

$$\text{or } p \propto \frac{1}{v}.$$

If (p_1, v_1), (p_2, v_2) are two sets of conditions of the gas, then both $(p_1 \times v_1)$ and $(p_2 \times v_2)$ will be equal to the same constant or

$$p_1v_1 = p_2v_2.$$

(c) Charles' Law

This law, which is also known as Gay-Lussac's Law, can be stated as follows:

The volume of a gas at constant pressure is proportional to its *absolute temperature* (see textbooks on Heat).

$$\text{Thus } v \propto T.$$

$$\text{But from Equation (1) } v \propto \frac{1}{p}$$

$$\text{thus } v \propto \frac{T}{p}$$

$$\text{or } \frac{pv}{T} = \text{constant} \qquad . \quad . \quad . \quad (2)$$

If the symbol R is used for this constant, then

$$\frac{pv}{T} = R. \qquad . \quad . \quad . \quad . \quad (3)$$

This quantity R, a constant for any experiment of this sort, depends on the mass of gas occupying the volume v, but if this mass is 1 gram-molecule, then R takes a specific value and is known as the *Universal Gas Constant.*

If (p_1, v_1, T_1) and (p_2, v_2, T_2) represent two sets of conditions of the same mass of gas, then:

$$\frac{p_1v_1}{T_1}=\frac{p_2v_2}{T_2}.$$

7.5 Ideal or Perfect Gas

A gas which obeys exactly the gas law expressed in Equation (3) is said to be a *Perfect* or *Ideal Gas*; in fact, no gas has been found to obey this law over all conditions of temperature and pressure, but it is found that all gases obey the law very closely over a small range centred on a specific temperature called the *Boyle Temperature* (each gas has its own value of Boyle Temperature). In this region the gas behaves as an ideal gas.

Further, it is found experimentally that the differences between the behaviour of various gases gets smaller as the pressure is lowered. We believe that the differences would vanish if the pressure in the gas were reduced to zero. This, of course, is impossible in practice, but measurements made on a gas at every low pressures give a close experimental approach to an ideal gas.

In elementary laboratory work it is common to treat all gases as ideal under normal conditions of temperature and pressure. The error introduced is usually much less than the experimental errors involved!

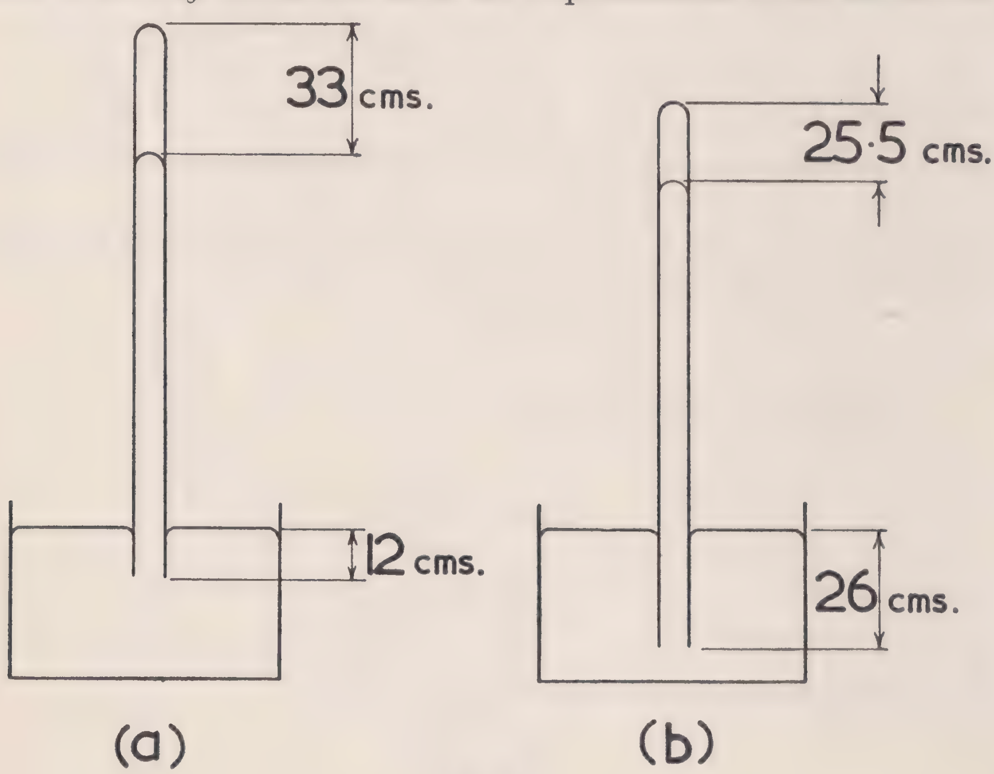


Fig. 7.1

Example 1. *A barometer tube* 100 *cm long is filled with mercury in the normal fashion and then a small quantity of air is admitted to the Torricellian vacuum. It is found that when the open end of the tube is submerged to various depths, the mercury levels are as shown in the diagram opposite. Find the atmospheric pressure.*

Work throughout in cm of mercury as the unit of pressure. Let the atmospheric pressure be h_0 cm of mercury, let the pressure above the mercury in case (*a*) be h_1 and in case (*b*) h_2 cm of mercury.

$$\text{Then from } (a)\ h_0 = h_1 + 55 \qquad \text{(i)}$$

$$\text{and from } (b)\ h_0 = h_2 + 48{\cdot}5 \qquad \text{(ii)}$$

$$\text{thus } h_1 + 55 = h_2 + 48{\cdot}5$$

$$\text{or } h_1 = h_2 - 6{\cdot}5 \qquad \text{(iii)}$$

Now the length of the air column is proportional to its volume if the tube is of constant bore, hence from Boyle's Law:

$$h_1 \times 33 = h_2 \times 25{\cdot}5$$

$$\text{or } h_2 = \frac{33h_1}{25{\cdot}5}.$$

Substituting this in Equation (iii) gives:

$$h_1 = \frac{33h_1}{25{\cdot}5} - 6{\cdot}5,$$

$$\text{thus } h_1\left(1 - \frac{33}{25{\cdot}5}\right) = -6{\cdot}5$$

$$\text{or } h_1 = \frac{6{\cdot}5}{0{\cdot}294}$$

$$= 22{\cdot}1 \text{ cm of mercury.}$$

Substituting this value in Equation (i) gives:

$$h_0 = 22{\cdot}1 + 55$$

$$= 77{\cdot}1 \text{ cm of mercury.}$$

7.6 Air Pumps

Air pumps may be used either to compress a gas into a container or to remove gas from a container. The latter are called vacuum pumps and are very important pieces of equipment in the modern research laboratory, where many experiments have to be performed in a vacuum. The reciprocating exhaust pump should be familiar to the reader and provides a simple introduction to the operation of a vacuum pump.

(a) The Exhaust Pump

The pump shown in Fig. 7.2 has valves so arranged that air is pumped out of the vessel A. As the piston is raised from the lowest position, the valve V_2 closes under its own weight and the motion of the piston then reduces the pressure of the air in B. This sets up a pressure difference across the valve V_2, since the upper part of the barrel is filled with air at atmospheric pressure, and hence presses the valve tightly shut.

As the expansion of B continues, the pressure will eventually fall to a value below that of the air in A; the pressure difference across the valve V_1 will now lift it open and air will pass from A to B.

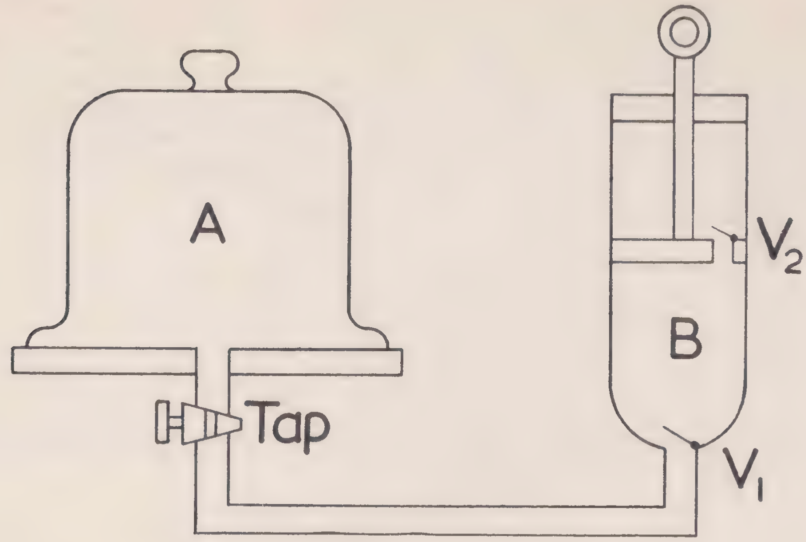


Fig. 7.2

When the piston reaches the top of its travel, the flow of air through V_1 ceases and the valve thereupon closes; the space B is now filled with gas at pressure rather less than atmospheric.

As the piston descends, the pressure in B increases until it exceeds the pressure of the atmosphere; V_2 then opens and the gas in B escapes to the atmosphere. The cycle is then repeated, removing more gas from the vessel A at each stroke.

Making some assumptions, we can produce a simplified theory of the pump as follows.

If the volume of the vessel being exhausted is V, the pressure in it originally is p, and if the volume of the barrel of the pump is v, then the gas in the vessel will be expanded to a volume $V + v$ during the up-stroke and the pressure will fall to p_1 where

$$pV = p_1 (V + v)$$

$$\text{or } p_1 = p\left(\frac{V}{V + v}\right).$$

This process is repeated during the next stroke, but the starting pressure for the cycle is p_1. If the finishing pressure is p_2, then as before:

$$p_2 = p_1\left(\frac{V}{V + v}\right)$$

$$= p\left(\frac{V}{V + v}\right)^2$$

and if the process is repeated for n strokes, then:

$$p_n = p\left(\frac{V}{V + v}\right)^n. \qquad (4)$$

The pressure in the vessel thus decreases in geometric progression, each stroke lowering it in the ratio $V/(V+v)$, and however long the pumping is continued the pressure will never become zero.

In this case the minimum attainable pressure is limited by the dead space below the piston, i.e. the space between the valves V_1 and V_2 when the piston is in its lowest position. If V_S represents the volume of this space while V_B is the volume swept by the piston during its stroke, then the compression ratio of the pump *on the downward stroke* is $V_S/(V_B + V_S)$. Now the exhaust valve V_2 will open only when the pressure in B exceeds 1 atmosphere, but on the downward stroke the gas in B can be compressed only in the ratio $(V_B + V_S)/V_S$, thus, if the pressure in A falls to a value less than (1/compression ratio), i.e. $V_S/(V_B + V_S)$ atmospheres, the pump will be unable to compress the gas trapped between V_1 and V_2 up to 1 atmosphere and so expel it through the exhaust valve.

With most reciprocating pumps of the type shown in Fig. 7.2, the compression ratio is about 8, so that the lowest pressure that they can attain is roughly 1/8 of an atmosphere or 10 cm of mercury.

(b) Rotary Vacuum Pumps

Pressures much lower than that mentioned in the paragraph above can be achieved with a rotary vacuum pump. One type of such a pump is shown in Fig. 7.3, but there are many modifications of this principle.

The rotor, mounted eccentrically in the cylinder, carries a pair of vanes which are pressed against the wall of the cylinder by a spring (and by centrifugal force when the pump is spinning).

As these vanes rotate, air is drawn from the vacuum system into the space A behind vane 1 (Fig. 7.3 (a) and (b)), and imprisoned in space C

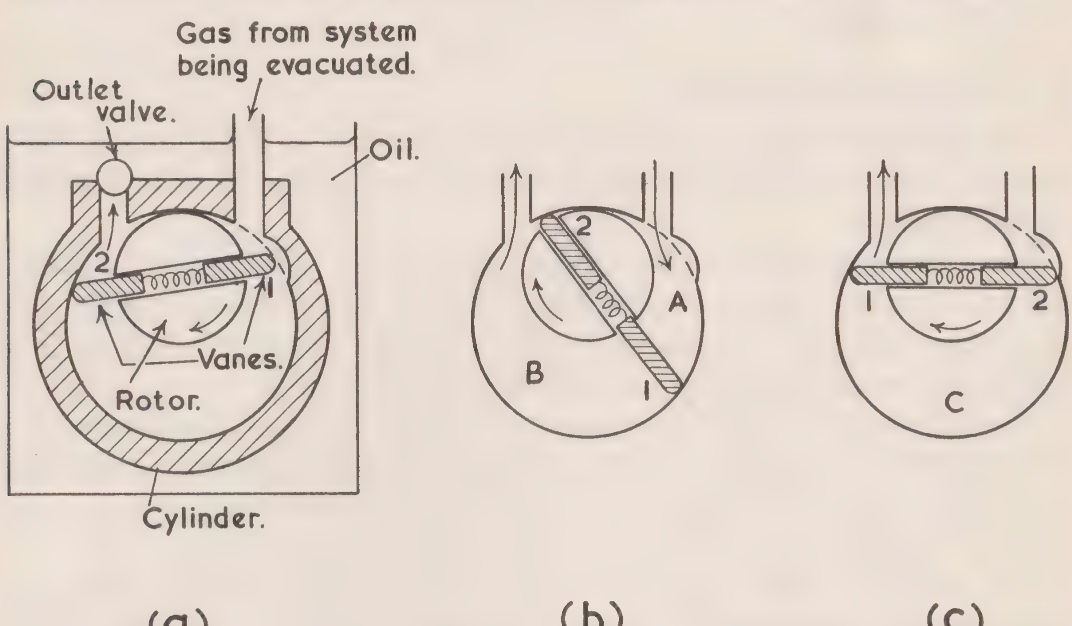


Fig. 7.3

(Fig. 7.3 (*c*)), when vane 2 passes the inlet port. This air is then swept round and expelled through the exhaust valve as in Fig. 7.3 (*a*), whereupon the cycle is repeated.

The theory of the exhaust pump described above can be applied to this pump, for it also expands gas from the vacuum system into the barrel of the pump (space A), compresses it until it exceeds atmospheric pressure, and then expels it from the exhaust valve. The pressure remaining in the vacuum system is given by Equation (4).

Just as before, the ultimate pressure produced by this pump is limited by the volume of the dead space—in this case the tube leading to the exhaust valve. The pump, however, is liberally lubricated and immersed in oil to make a good airtight seal between the vanes and the wall and at all the joints in the system. Some oil is drawn in at each stroke and carried round by the vanes, finally being expelled through the exhaust valves. The dead space is thus filled with incompressible oil and its size reduced practically to zero in a well-designed pump. With such a pump it is possible to produce a pressure of 0·00001 atmosphere or roughly 0·01 mm of mercury. A pressure as low as this is usually called a 'vacuum', and we should say that the 'ultimate vacuum of the pump is 0·01 mm of mercury'. Note that the 'mm of mercury' is the unit normally used for measuring the pressure in a vacuum system.

7.7 Vacuum Gauges

Various devices for measuring the pressure of a gas have been described in the previous chapter, page 163. Some of them, for example the bellows manometer and the closed-limb U-tube, work down to very low pressures, but special gauges are used to measure the pressure in a vacuum system.

(a) The McLeod Gauge

The *McLeod Gauge* is used for low-pressure measurements. It is normally regarded as a standard and used to calibrate other forms of vacuum gauges.

The basic principle of this gauge, seen in Fig. 7.4, is that it takes a known volume of gas at a pressure so low as to be unmeasurable, then compresses the gas in a known ratio until the pressure becomes large enough to be measured by an ordinary manometer.

The top of the tube T is connected to the vacuum system which is then pumped down to a very low pressure p. This means that the mercury in the tube L will stand at about the barometric height above the level of the mercury in the reservoir R, Fig. 7.4 (*a*).

The height of the reservoir is so adjusted that initially the mercury level in L is below the level of the side tube at A so that the whole of the bulb B is filled with gas at the low pressure p. The reservoir is

now raised; this lifts the meniscus in L and imprisons a volume V of gas in the bulb by sealing off the side tube at A. Further raising of the reservoir compresses the gas into the capillary tube sealed to the top of the bulb as in Fig. 7.4 (b).

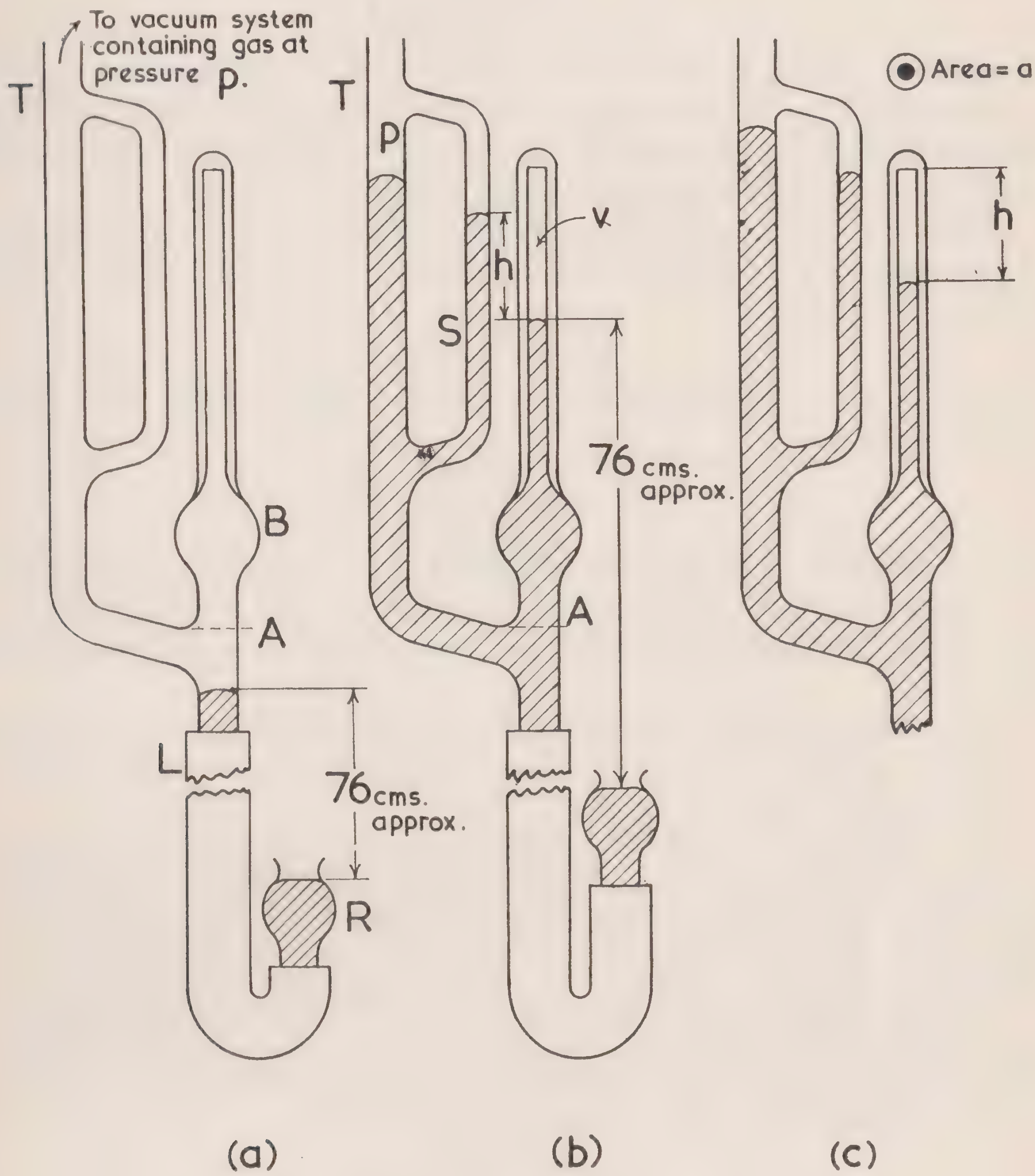


Fig. 7.4

Suppose that the volume of the gas in the capillary tube is now v and its pressure P, then from Boyle's Law

$$pV = Pv.$$

The mercury will now be found to stand h cm higher in the side tube S than in the capillary tube. Considering these as the two arms of a U-tube manometer, we see that:

$$P = p + h \quad . \quad . \quad . \quad . \quad (5)$$

if the pressure is measured in centimetres of mercury. Substituting this value for P in the equation above gives:

$$pV = (p + h)v$$

$$\text{or } p = h \cdot \left(\frac{v}{V + v}\right) \qquad . \quad . \quad . \quad (6)$$

Thus if the volume V of the bulb is known and if the capillary is calibrated so that the volume occupied by the air after compression can be measured, then the pressure of the gas can be calculated; the instrument, however, is usually made direct-reading as follows.

If the mercury is raised until the meniscus in S is level with the top of the capillary tube (which is of uniform bore and square ended as in Fig. 7.4 (*c*)), then:

$$v = ah$$

where a is the cross-sectional area of the capillary tube, thus from Equation (6):

$$p = h \cdot \left(\frac{ah}{V + ah}\right) \qquad . \quad . \quad . \quad (7)$$

This equation involves only the variable h, for V and a are constants of the instrument, thus a scale can be constructed and fixed against the capillary tube having the pressure engraved against corresponding distances h.

The normal pressure range covered by the McLeod gauge is from 10 mm of mercury down to 10^{-5} mm—several gauges of different size would be needed for this range.

Any errors which may be expected due to the effects of surface tension (see Chapter 9) are avoided by making the tube S of the same bore as the capillary, thus the same error appears on each side of the manometer and cancels out.

The McLeod gauge provides an interesting example of the behaviour of a vapour when compressed. Suppose that the pressure in the vacuum system is made up of p_1 due to a gas and p_2 due to a saturated vapour, then the total pressure p is given by:

$$p = p_1 + p_2 \qquad . \quad . \quad . \quad . \quad (8)$$

Now Dalton's Law indicates that each of these can be treated separately, thus when the *gas* is compressed from volume V to v, its pressure will go up to P_1, where:

$$p_1V = P_1v$$

$$\text{or } P_1 = p_1\frac{V}{v}$$

When the *vapour* is compressed, however, it will liquefy (the liquid occupying negligible volume) and maintain its saturation vapour pressure p_2.

The total pressure in the capillary after compression will thus be given by:

$$\begin{aligned} P &= P_1 + p_2 \\ &= p_1 \frac{V}{v} + p_2 \quad . \quad . \quad . \quad . \quad (9) \end{aligned}$$

Substituting for p and P from Equation (8) and (9) in Equation (5) gives:

$$p_1 \frac{V}{v} + p_2 = p_1 + p_2 + h$$

$$\text{or } p_1 \frac{V}{v} = (p_1 + h)$$

which leads to:

$$p_1 = h\left(\frac{v}{V + v}\right),$$

$$\text{or } p_1 = h\left(\frac{ah}{V + ah}\right)$$

instead of Equations (6) and (7) respectively. The McLeod gauge thus measures only the partial pressure due to gases in the vacuum system and the pressure due to any saturated vapours is eliminated.

(b) Pirani Gauge

One of the drawbacks in the use of the McLeod gauge is the fact that it does not give a continuous reading of pressure. A sample of gas is collected and its pressure measured, but if the pressure in the vacuum system changes subsequently, the McLeod gauge does not record the variation until another sample can be collected. Many other gauges have been devised to overcome this trouble, but they all rely on very indirect measuring processes and have to be calibrated against a McLeod gauge.

It is found that at low pressures the rate of loss of heat from a very fine hot wire decreases with the pressure of the gas surrounding the wire. If the wire is heated electrically, the current needed to maintain it at a given temperature will vary with the gas pressure. This is the basic principle of the *Pirani Gauge* which will work from a pressure of 10 mm down to 10^{-4} mm of mercury.

In practice, the hot wire of the Pirani gauge is a 'hair-pin' of wire mounted in a glass envelope similar to a radio valve but provided with a ground glass joint so that it can be 'plugged-in' to a vacuum system.

The resistance of a wire changes with temperature, and so it can be maintained at a fixed temperature by keeping its resistance constant. This is done by making the hot wire one arm of a Wheatstone bridge, the resistance of the other arms being chosen so that the bridge is balanced when the wire has the correct resistance (Fig. 7.5).

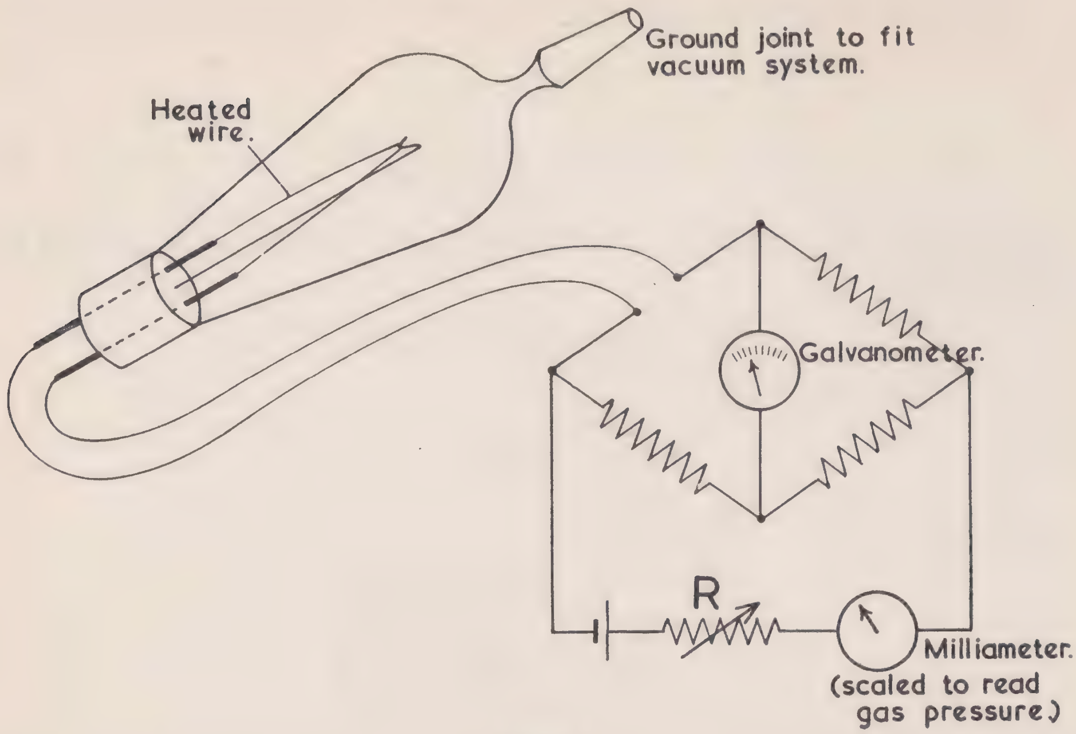


Fig. 7.5

If the galvanometer of the bridge shows a deflection either to the right or left, it indicates that the wire is too hot or too cold; the heating current is then adjusted by the rheostat R until the galvanometer shows no deflection.

The current, as explained above, varies with pressure. A milliammeter in series, with R will record the heating current, but more usually it is scaled to read directly in terms of pressure. The connection between current and pressure is too complicated for any calculation of the relationship to be made; instead, the instrument is always calibrated by the manufacturers against a McLeod gauge.

(c) Ionisation Gauge

The other main type of gauge in use is the *Ionisation Gauge*. At ordinary pressures gases are quite good electrical insulators, but as the pressure is decreased, their insulating properties deteriorate only to improve again at very low pressures. Thus if a constant voltage is applied between two electrodes in a gas at low pressure, the current flowing between them will depend on the pressure and can be used to measure it. The current is very small and is recorded on a microammeter which is scaled to read pressure directly by calibration against a McLeod gauge. Various types of ionisation gauge are available covering a pressure range of 1 mm down to 10^{-8} mm of mercury.

The pressure ranges covered by the gauges described above and in Chapter 6 are summarised in diagrammatic form in Fig. 7.6.

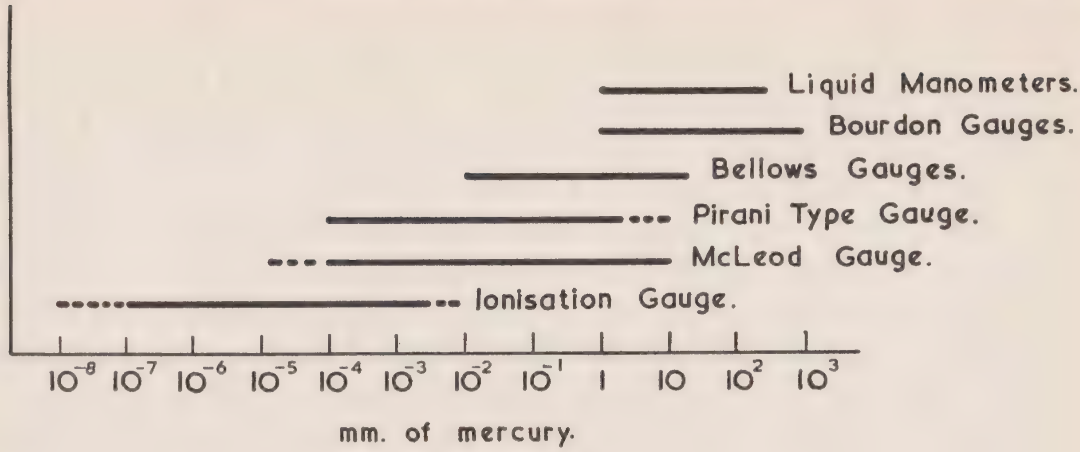


Fig. 7.6

7.8 Isothermal and Adiabatic Conditions

It is common experience (derived usually from the use of a bicycle pump) that a gas when compressed becomes hot. For theoretical purposes, however, it is often assumed that this heating can be avoided by carrying out the compression so slowly that the heat can be removed by a cooling system as fast as it is generated. If this could be done, then the gas would be compressed at a constant temperature or *Isothermally*.

If the heat is not removed, the condition is said to be *Adiabatic* and the gas is found to obey the law

$$pv^{\gamma} = \text{constant} \quad . \quad . \quad . \quad . \quad (10)$$

The exponent γ is a constant for a particular gas and usually has a value between 1 and 2.

A more detailed treatment of Equation (10) and a discussion of the theoretical significance of γ will be found in *Heat*, by Woodall, published by English Universities Press, Ltd.

7.9 Kinetic Theory of Matter

Many attempts have been made to explain the properties of matter in terms of the mechanical behaviour of molecules, which are supposed to be in a perpetual state of agitation. This motion can be made to account for many of the physical properties of matter.

The kinetic theory suggests that the difference between the states of matter is due merely to the freedom of the molecules to move. Thus, in a solid, the molecules are packed closely in a regular geometric pattern and are bound together with forces sufficiently strong to prevent any movement greater than a vibration of each molecule about its fixed position in the pattern. If a solid could be so enlarged that the individual molecules were visible, then it would appear as a quivering mass of molecules instead of a rigid body.

The kinetic energy of the vibrating molecules is regarded as representing the heat energy possessed by the body, so that as a solid is warmed, its increasing heat energy appears as a more and more violent

vibration of the molecules. Finally, the vibration becomes so violent that the molecules break away from their fixed positions in the pattern and move about, sometimes joining up with a few others to form a pattern and then immediately breaking away again; the matter is then said to be in the liquid form. If a liquid could be seen on an enlarged scale, the random wandering of the molecules would be visible. In addition it would be noticed that those molecules near the surface of the liquid appear still to be bound quite strongly together. The other molecules are prevented from leaving the liquid in any great numbers by this surface layer. This point is discussed in more detail in Chapter 9.

Finally, if a liquid is heated, the motion of the molecules becomes sufficiently violent for them to break through the surface layer of molecules in large numbers into the space beyond; the matter is then said to be in the gaseous state. In this condition the molecules are very widely scattered, the distance between them being so large that they exert very little influence on each other. The vibrating motion has now vanished, and instead, the molecules travel along straight paths uninterrupted for a short time, collide with other molecules and then move off in different directions for another short period.

Fig. 7.7

The separation of molecules, relative to their size, in the three states of matter, is shown in Fig. 7.7.

The kinetic theory has shown some success in the treatment of the solid state, where the molecules are all in known positions and the forces between them can be calculated. It is especially successful in the treatment of the gaseous state where the molecules have so little interaction with each other that they can be treated as individual particles; but the liquid state has not proved so amenable, and we still have no really successful kinetic theory of the liquid state.

7.10 Kinetic Theory of Gases—Pressure of a Gas

The first phenomenon exhibited by a gas to be explained by the motion of the molecules is the pressure exerted by a gas on the walls of the containing vessel. If all the molecules of the gas are in motion, then they will make frequent collisions with the walls, and this continual bombardment of the walls will be equivalent to a pressure exerted on the walls.

In the mathematical treatment of the kinetic theory some simplifying assumptions must be made at this stage; they will be introduced as necessary in the work, but it must be remembered that more advanced theories do exist in which these assumptions are progressively removed.

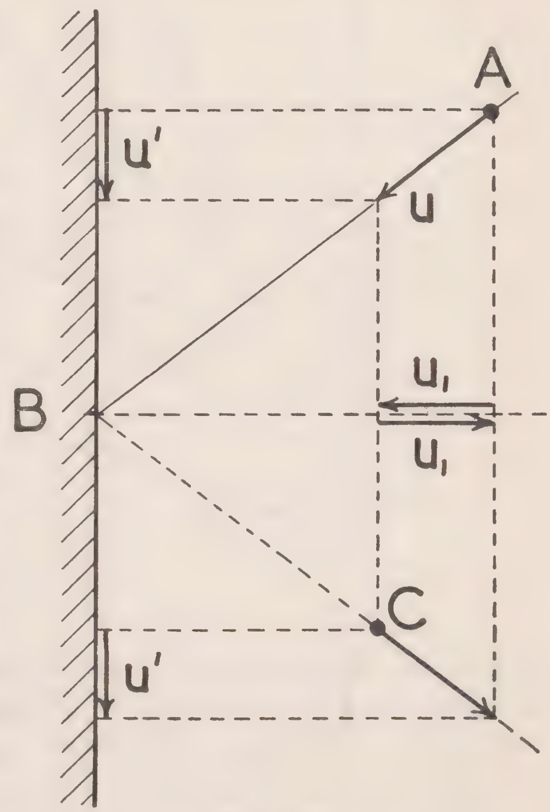


Fig. 7.8

First of all, assume that all molecules in an enclosed space have the same mass m. If one of these molecules collides with the wall (molecule and wall both assumed to be perfectly smooth), and if the impact is perfectly elastic, its path will be ABC as shown in Fig. 7.8.

On impact, its velocity u will have two components, u_1 perpendicular to the wall and u' parallel to the wall; after impact, u' will remain unchanged while u_1 will be reversed in direction. The molecule thus suffers a change in momentum $2mu_1$ perpendicular to wall (mu_1 *towards* the wall before impact and mu_1 *away* from the wall after impact) and no change in the direction parallel to the wall.

If all the molecules had the same velocity u_1 perpendicular to the wall it would now be easy to calculate the rate of change of momentum, and hence the force exerted on the wall; unfortunately we cannot make this simple assumption and must proceed as follows.

Let unit volume of the gas contain n_1 molecules, having a component of velocity u_1 perpendicular to the wall; this group of molecules will be referred to as G_1. Similarly G_2 will be a group of n_2 molecules having a velocity u_2 perpendicular to the wall, etc. Consider first G_1 and imagine a cylindrical volume of cross-sectional area A and height $u_1 . \delta t$ as shown in Fig. 7.9 with the end marked A adjoining the wall.

Each molecule moves a distance $u_1\delta t$ perpendicular to the wall in time δt. Thus, if a molecule finds itself within this cylinder and moving towards the wall, it is bound to hit the wall within the time interval δt. If it is moving away from the wall, of course, no impact will

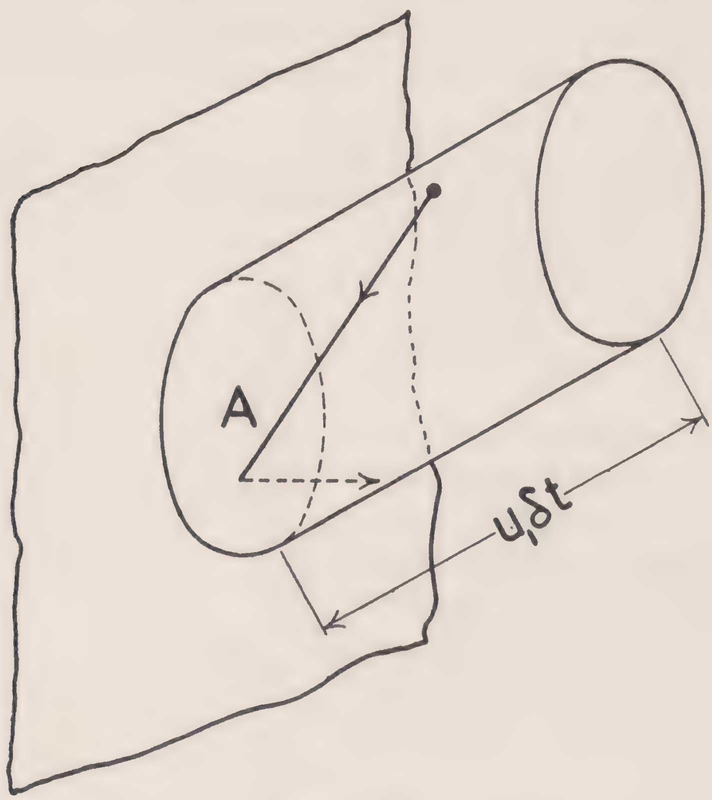


Fig. 7.9

take place. On the average half of the molecules will be moving in each direction, and so the number of molecules in G_1 moving towards the wall can be calculated as follows: volume of cylinder $= Au_1\,\delta t$,

but number of molecules per unit volume $= n_1$

thus the number of molecules in cylinder $= n_1Au_1\,\delta t$ (10)

and number of molecules moving towards wall $= \frac{1}{2}n_1Au_1\,\delta t$. (11)

Each one of these molecules produces an impact within the interval δt and suffers a change in momentum of $2mu_1$, therefore the change in momentum δM produced in time δt is given by:

$$\delta M = 2mu_1 \cdot \frac{n_1}{2} Au_1\,\delta t$$

$$\text{or } \delta M/\delta t = An_1mu_1^2.$$

In the limit, when $\delta t \to 0$, this becomes:

$$dM/dt = An_1mu_1^2 \quad . \quad . \quad . \quad . \quad . \quad (12)$$

But dM/dt is the rate of change of momentum which is a force; therefore the force exerted on the wall by these molecules is given by:

$$F = An_1mu_1^2 \quad . \quad . \quad . \quad . \quad . \quad (13)$$

It will be noticed that all the molecules starting within the cylinder of Fig. 7.9 do not hit the area A since u_1 is only the component of their velocity perpendicular to the wall; they may have another component which will cause them to move obliquely out of the cylinder. The chance of this happening will be less and less, however, as the height of the cylinder is reduced, i.e. as δt grows smaller, and when $\delta t \to 0$, all the molecules starting within the cylinder will hit the area A. Alternatively, we may argue that on the average as many molecules will enter the cylinder as leave it, consequently the result will be the same as if all the molecules starting within the cylinder hit the area A.

The force F derived in Equation (13) can thus be considered to be applied over the area A, or the pressure on A is given by:

$$p_1 = \frac{F}{A}$$

$$= n_1 m u_1^2 \qquad \text{(14)}$$

In a similar fashion, the pressure produced by the G_2, G_3, etc., groups of molecules can be calculated, they will contribute pressures p_2, p_3, etc., given by:

$$p_2 = n_2 m u_2^2$$

$$p_3 = n_3 m u_3^2, \text{ etc.}$$

Hence the total pressure p is given by:

$$p = p_1 + p_2 + p_3 + \ldots$$

$$= m(n_1 u_1^2 + n_2 u_2^2 + n_3 u_3^2 + \ldots) \qquad \text{(15)}$$

Now at this stage of the theory it is quite impossible to discover how the velocities of the molecules are distributed, i.e. to find how many molecules have the velocity u_1, etc. The simplest procedure is to take some average value of velocity, and since Equation (15) indicates that the pressure depends on the square of the velocity, we take an average value of (velocity squared). Assume, then, that:

$$n_1 u_1^2 + n_2 u_2^2 + n_3 u_3^2 + \ldots = n\overline{u^2}$$

where n is the total number of molecules per unit volume, i.e.

$$n = n_1 + n_2 + n_3 + \ldots,$$

$$\text{thus } \overline{u^2} = \frac{n_1 u_1^2 + n_2 u_2^2 + n_3 u_3^2 + \ldots}{n_1 + n_2 + n_3 + \ldots}$$

The symbol $\overline{u^2}$ indicates the average value of all the velocities *after* they have been squared and is called the '*mean square velocity*'. Making this substitution in Equation (15) gives:

$$p = nm\overline{u^2} \qquad \text{(16)}$$

This equation gives the pressure in terms of the molecular mean square velocity perpendicular to one wall, but it would be better if the pressure were expressed in terms of the true molecular velocity.

Let the true velocity c of any molecule have components u, v, w, along three mutually perpendicular axes, then:

$$c^2 = u^2 + v^2 + w^2.$$

But it is known that any one direction in the gas is no more favoured than another, for the gas exerts the same pressure on all the walls whatever their direction, thus $\overline{u^2}$, $\overline{v^2}$ and $\overline{w^2}$ will all be equal, and:

$$\overline{c^2} = \overline{u^2} + \overline{v^2} + \overline{w^2}.$$
$$= 3\overline{u^2}.$$

Substituting this in Equation (16) gives:

$$p = \tfrac{1}{3}nm\overline{c^2} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

This expression can be written in several alternative forms; for example, m is the mass of 1 molecule and n the number of molecules per unit volume, thus mn is the mass of unit volume of the gas, which is equal to ϱ, its density. Hence:

$$p = \tfrac{1}{3}\varrho\overline{c^2} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Also Equation (17) can be written as:

$$p = \tfrac{2}{3}n(\tfrac{1}{2}m\overline{c^2}).$$

But $(\frac{1}{2}m\overline{c^2})$ is the mean kinetic energy of one molecule and $n(\frac{1}{2}m\overline{c^2})$ is the mean kinetic energy of all the molecules in unit volume, thus:

$$p = \tfrac{2}{3}\text{ (mean kinetic energy of molecules in unit volume)} \quad . \quad (19)$$

7.11 Derivation of the Gas Laws from the Kinetic Theory
(a) Dalton's Law

If, instead of dealing with a pure gas whose molecules are all of mass m, we consider a mixture of gases containing:

n_1 molecules of mass m_1 and mean square velocity $\overline{c_1^2}$ per unit volume,
n_2 molecules of mass m_2 and mean square velocity $\overline{c_2^2}$ per unit volume,
etc., etc.

Then the bombardment of the walls by each group of molecules will give rise to pressures p_1, p_2, etc., where:

$$p_1 = \tfrac{1}{3}n_1m_1\overline{c_1^2}$$
$$p_2 = \tfrac{1}{3}n_2m_2\overline{c_2^2}, \text{ etc.}$$

The total pressure p is the sum of each of these pressures, thus:

$$p = p_1 + p_2 + \cdots$$

Now p_1, p_2, etc., are the pressures which each gas would exert if it alone occupied the space, hence this equation is merely a mathematical statement of Dalton's Law of Partial Pressures.

(b) Boyle's Law

Consider next any volume V of a gas and multiply each side of Equation (17) by V, giving:

$$pV = \tfrac{1}{3}nV \, . \, m\overline{c^2}.$$

The factor (nV) is the number of molecules in the volume V; write this as N, hence

$$pV = \tfrac{1}{3}N \, . \, m\overline{c^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Now it was suggested earlier that the heat energy in a gas is represented by the motion of the molecules. If this is so, then at a constant temperature we should expect the kinetic energy ($\tfrac{1}{2}m\overline{c^2}$) of the molecules to be constant, also N, the number of molecules in an enclosed space must of necessity be constant. Inserting these conditions into Equation (20) leads to:

$$pV = \text{constant},$$

which is recognised as Boyle's Law.

The fact that the assumption of a relationship between temperature and the kinetic energy of molecules leads to a well-established law does not prove that the assumption is correct; it does, however, encourage us to seek further confirmation. This idea has been extended until now most of the properties of a gas can be explained in terms of the kinetic energies of the molecules of the gas. For further details the student is referred to *Heat*, by Woodall, published by English Universities Press, Ltd.

EXERCISES 7

1. A cylindrical diving bell of internal radius 3 ft. and internal height $6\frac{2}{3}$ ft. is lowered into water until its base is 30 ft. below the water-level. Find the height to which the water rises inside the bell if the height of the water barometer is 33 ft.

   Find also the volume of air, at atmospheric pressure, that must be pumped into the bell in order to exclude the water.

   (London Univ. Inter. B.Sc.)

2. On what factors does the pressure at a point in a fluid depend? How would you show that this pressure has the same magnitude in all directions at a constant depth in a static fluid?

   At what depth in water would an air bubble just fail to rise? Assume that the temperature remains constant, that the air obeys Boyle's Law and that water is incompressible.

   (Density of air at standard pressure = 1·35 gm./litre: 1 atmosphere = 10^6dynes/cm.^{2}) (Cambridge H.S.C.)

3. Discuss the factors which determine the equilibrium of a floating body.

   A cylindrical glass tube, of mass 8·15 gm., length 10·0 cm., and internal area of cross-section 1·5 cm.^{2}, is closed at one end. It is two-thirds filled with water, and then immersed, open end downwards, in a

deep vessel of water, where it floats almost completely submerged with its axis vertical. Show that it will return to this position if it is given a very small vertical displacement, and find the least depth below the free surface of the water to which the closed end of the tube must be pushed down in order that the tube, when released, shall continue to sink.

(Take the atmospheric pressure as 10^3 gm.wt. per sq.cm. and the density of glass as 2·5 gm.cm.$^{-3}$) (Oxford H.S.C.)

4. State Boyle's law, pointing out the conditions under which it may be applied to an ordinary gas.

A capillary tube containing air is sealed at one end and is closed at the other end by a short thread of mercury. When the tube is held vertically with the open end up, the length of the enclosed air is 10 cm. and on reversing the tube (open end down) the length of the air is 15 cm. Find the length when the tube is held horizontally. (Manchester Univ. Schol.)

5. A barometer is known to have a small quantity of air above the mercury. When the top of the tube is 80 cm. above the level of the mercury in the reservoir, the height of the mercury column is 74·5 cm. The tube is then depressed until its top is 75 cm. above the level in the reservoir and the height of the column is found to fall to 74 cm. Find the pressure of the atmosphere. (Cambridge Univ. Schol., King's College Group.)

6. What is meant by the pressure at a point in a fluid? Describe how you would measure (*a*) a gas pressure of the order of 10 atmospheres, (*b*) a gas pressure of the order of 10^{-3} atmospheres.

A vacuum pump with volume 200 c.c. is used to exhaust a vessel of volume 2,500 c.c., in which the initial pressure is 75 cm. of mercury. Find the pressure in the vessel after 25 strokes of the pump. (Oxford H.S.C.)

7. Describe the Bourdon gauge for measuring high pressures, and explain how it is used.

Give an account of experiments which have been made to investigate the deviation of gases from Boyle's law at high pressures, and discuss briefly the results observed in such experiments. (Oxford G.C.E., Advanced level.)

8. Describe some form of air pump for producing a high vacuum, and explain upon what principles its action depends.

A certain air pump takes in a fixed volume of air V, measured at the intake pressure, at each stroke. The air is compressed isothermally to atmospheric pressure P, when the outlet valve opens and the air is expelled. Show that at a certain intake pressure the work done per stroke is a maximum and is equal to PV/e, where e = base of Naperian logarithms. (Oxford Univ. Schol.)

9. Give an outline of the design, and describe the mode of action of a high-vacuum pump.

Gas is pumped continuously from a 'leaky' container of 10 litres capacity, and the lowest pressure reached is 1 mm. of mercury. The

pump removes gas at the rate of 30 litres per min. If the pump is stopped, what will be the initial rate of rise of pressure in the container? (Isothermal conditions may be assumed.)

(Cambridge G.C.E. Advanced level)

10. Calculate the work which must be done in order to compress 50 litres of air at atmospheric pressure isothermally to a volume of 2·5 litres.

(Take the atmospheric pressure to be 10^6 dynes per sq.cm. and $\log_e 10$ to be 2·303.)

Oxygen cylinders are tested by filling them with water to a pressure P, which is considerably greater than the pressure which they must withstand when filled with gas. Explain why this is a much safer procedure than testing them with compressed air to the pressure P.

(Oxford G.C.E. Schol. level.)

11. How are the gas laws accounted for by the kinetic theory of gases? Calculate from the following data the root mean square velocity of a hydrogen molecule at a temperature 100° C.

Density of hydrogen at N.T.P. = 0·090 grams per litre.

Density of mercury = 13·6 grams per cc. (Oxford Univ. Schol.)

12. What interpretation does the kinetic theory give of (*a*) the pressure, (*b*) the temperature, (*c*) the heat content of a gas?

Calculate the r.m.s. velocity of molecules of oxygen at a temperature of 17° C. (Oxford Univ. Schol.)

13. On the basis of the kinetic theory deduce an expression for the velocity of the molecules of a gas in terms of quantities which determine the state of the gas. Point out the simplifying assumptions which you make in your calculation, and indicate the effect of the corrections necessary if these assumptions are not made.

Calculate the velocity of oxygen molecules under normal conditions of an experiment in the laboratory. If these molecules have a radius of gyration of 10^{-8} cm., with what angular velocity will they be rotating? (Cambridge Univ. Schol., King's College Group.)

CHAPTER 8

ELASTICITY

8.1 Introduction

When developing the simple theory of mechanics, bodies are very often described as 'rigid', meaning that if they are subjected to various forces, the shape and size of the body remains unaltered. In practice, of course, there is no such thing as a 'rigid body'—everything changes its shape slightly when subjected to a force and in this chapter the nature of these changes will be studied.

8.2 Elastic Behaviour of a Stretched Wire

(a) Hooke's Law

As an introduction, consider the effect upon the length of a well-annealed piece of wire when it is subjected to a gradually increasing stretching force. The curve shown in Fig. 8.1, in which the extension of the wire is plotted against the stretching load, is a typical case.

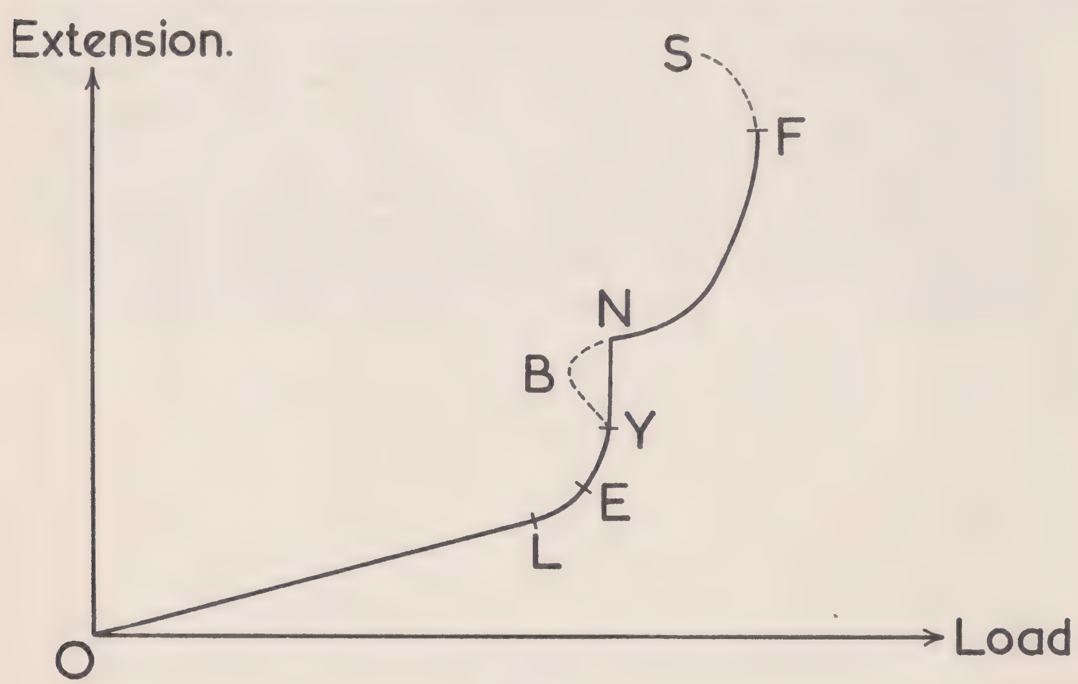


Fig. 8.1

It will be seen that, as the load increases steadily from zero, the extension at first increases in proportion to the load; this is represented by the region *OL* of the curve.

Over this range the extension is proportioned to the load; this fact was recognised by Hooke in 1678 and is known as *Hooke's Law*.

(b) Limit of Proportionality

The end of the straight region of the curve (the point L in the diagram) is called the *Limit of Proportionality*, since beyond this point the graph is no longer a straight line and hence the extension is no longer proportional to the load.

(c) Elastic Limit

If the wire is stretched to the point L and then the load decreased again, it is found that the wire returns to its original length and the curve is traced out in the reverse direction from L to O. In some materials this property holds good even for extensions beyond L, but fails at a point such as E. This point is called the *Elastic Limit*, and over the range O to E the wire is said to be *perfectly elastic*, even though at the extreme end of this range (L to E) Hooke's Law is not obeyed.

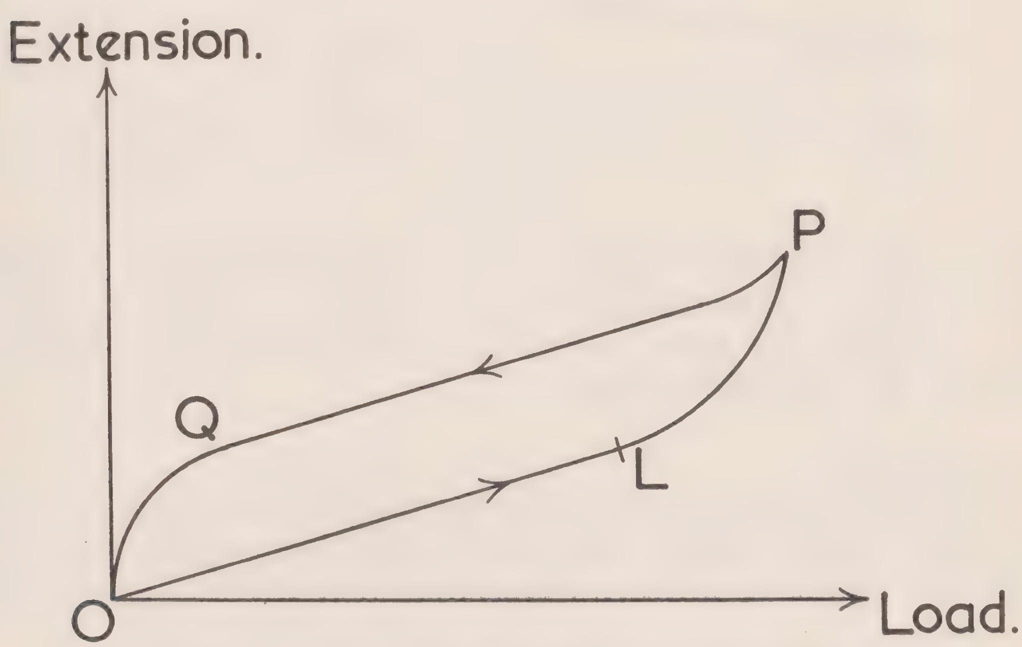


Fig. 8.2

The criterion of 'perfect elasticity' is that the extension–load diagram for decreasing load should follow exactly the curve for increasing load, and not that the wire should regain its original length when the load is removed. Many materials are known (nylon fibre is an example) for which the curves are as shown in Fig. 8.2. On stretching, the extension of the material is represented by the curve OLP, but it follows the curve PQO when the tension is relaxed. It will be seen later in this chapter that a curve forming an open loop indicates a loss of energy during the stretching and relaxing process; consequently the material is not described as perfectly elastic, even though it does return to its original length.

If the wire is stretched considerably beyond the elastic limit, then,

on removing the load the wire does not return to its original length, but contracts along a curve such as AA' (Fig. 8.3) and suffers a permanent extension OA'.

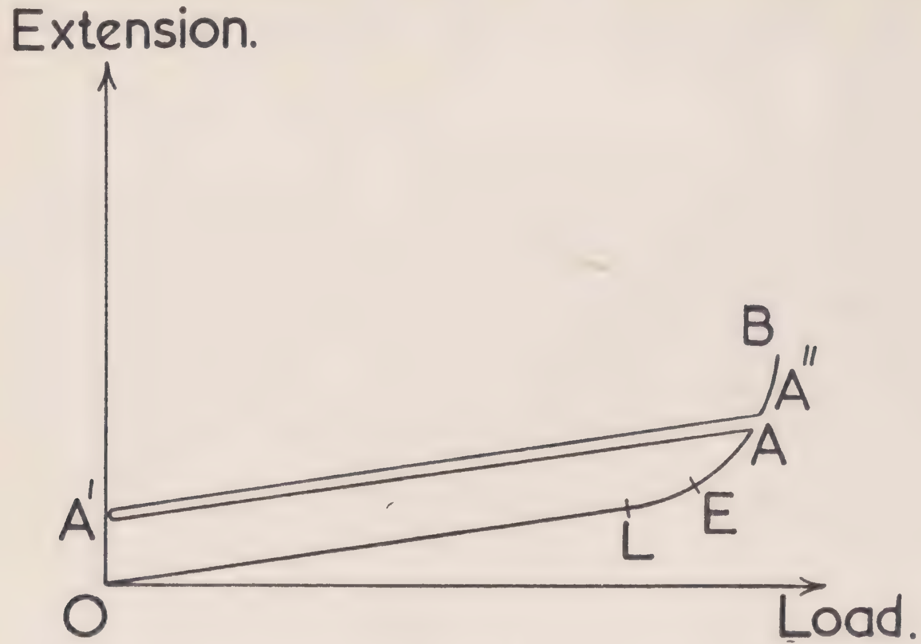


Fig. 8.3

If the load is once more applied, the wire now stretches along the line $A'A''$ and then follows the original curve from A'' to B. Over the region $A'A''$ the wire once again behaves in a perfectly elastic fashion, although it has been overstrained.

(d) Yield Point

If the wire is stretched still farther beyond the elastic limit the curve gradually becomes steeper; equal increments of load produce larger and larger increments of extension, indicating that the wire is becoming

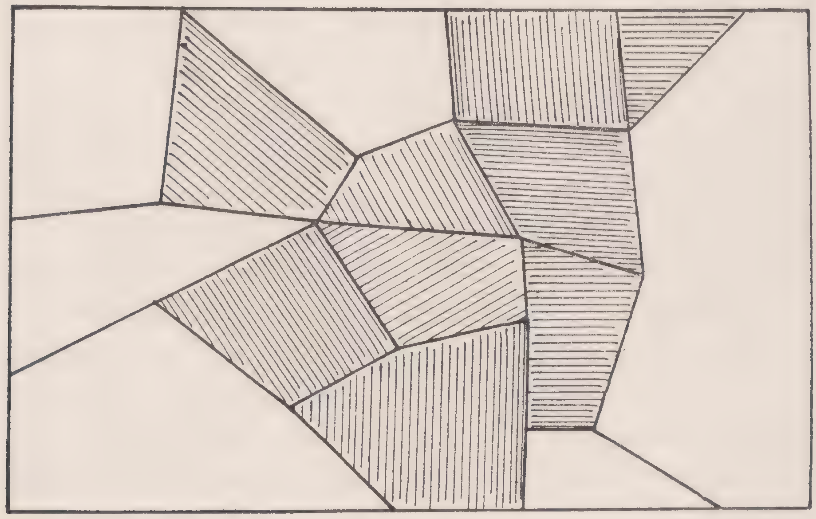


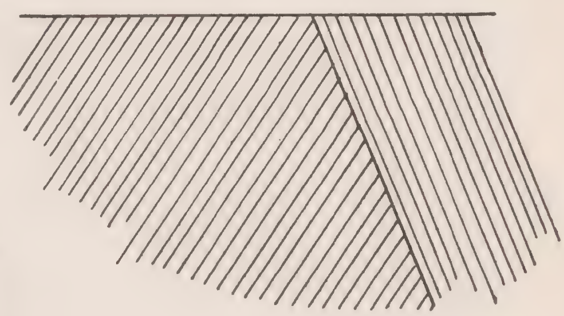
Fig. 8.4

weaker. Finally the curve becomes vertical at Y (Fig. 8.1), called the *Yield Point,* and a large extension (to the point N) occurs with no increase in load. If the stretching is done by a machine which can quickly release the load as the wire starts to give, the large extension can be produced with a momentarily *reduced* load as shown by the curve YBN.

At the yield point, a marked change appears in the material of the wire, its surface becoming dull and rough. Recent work has shown that metals are crystalline in nature, with the crystals fitted together in islands. The crystals are packed in different directions in each island (shown highly magnified in Fig. 8.4).

When the wire is strained beyond the yield point, the crystals in each of the islands slip along the planes in which they are packed so that a surface originally smooth as in Fig. 8.5 (*a*) becomes rough as in Fig. 8.5 (*b*)—again highly magnified.

At the yield point, a sudden and large slip of the crystal planes takes place and this rearranges some of the smaller islands of crystals so that they fit together better; this strengthens the material and the flow stops. The material does not, however, regain its original strength and is in a weaker state after yielding.



(*a*)

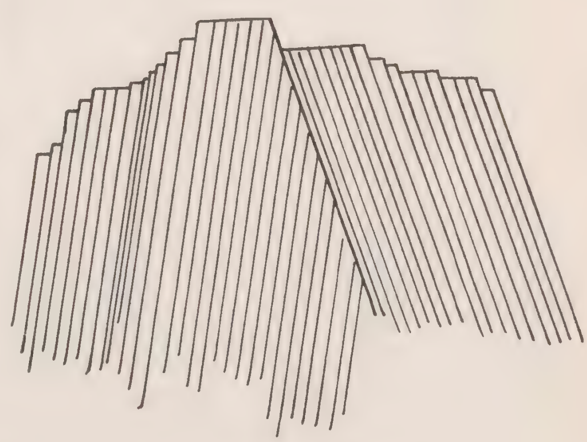


Fig. 8.5 (*b*)

(e) Plastic Flow

After a wire has been stretched beyond the yield point, the extension continues to increase with time, even though the load remains constant; similarly Plasticine will continue to yield indefinitely if a force is applied to it. In this condition the material is described as plastic and it is said to undergo *plastic deformation.*

Plastic flow produces a permanent deformation of the body and causes a gradual thinning of the wire throughout its length, its strength thus decreases and the extension-load graph curves upwards again beyond the yield point. Eventually the thinning of the wire develops into local 'necks', causing sudden weakening, the wire increases rapidly

in length (point F on the curve of Fig. 8.1, called the *Flow Point*) and, after a large extension, breaks at the *Breaking Point, S*.

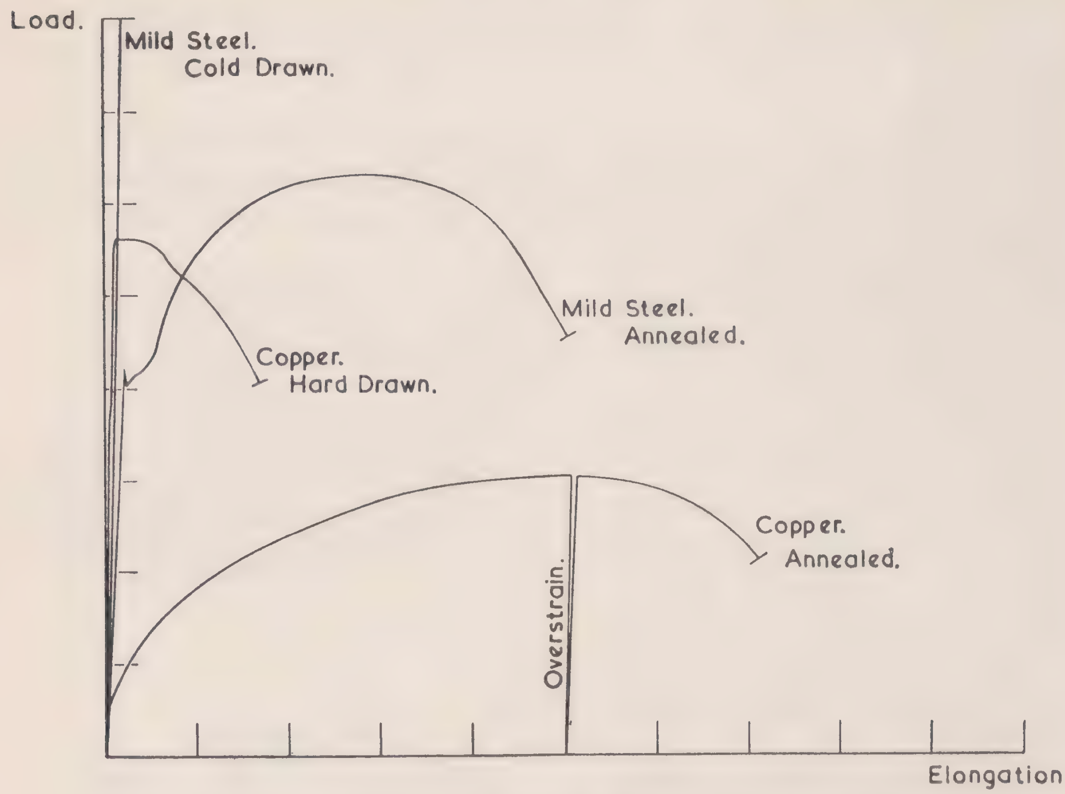


Fig. 8.6

The curve shown in Fig. 8.1 is not obtained for every material. All of the major points noted are invariably represented, but sometimes two or three of them coincide and this appears to change the shape of the curve considerably. It is most common for the limit of proportionality, elastic limit and yield point all to crowd together. Some typical curves are shown in Fig. 8.6, but it will be noticed that the axes of this graph are reversed in position from Fig. 8.1; this is in common with engineering practice (although it does not follow the usual method of representing the independent variable along the horizontal axis). Presenting the curves in this fashion means, as will be seen later, that the work done in stretching a wire can be found from the area underneath the graph. Moreover, a stronger wire results in a steeper curve, which is perhaps rather more graphic.

Wires normally encountered in the laboratory are not in the annealed state—they are formed by drawing down from a solid rod through a circular die. In the process the wire is strained beyond the yield point Y to a point A (Fig. 8.7) as it passes through the die; as it emerges it then relaxes to a point A'. The wire is in this condition when we commence our experiments on it, and as the wire is loaded, its extension is represented by $A'A''$, joining the original curve at A''.

It proceeds to the flow point F and the breaking point B, without apparently passing through a yield point.

The stretching of a wire has been described here as it is the most convenient shape on which to experiment; the results are applicable,

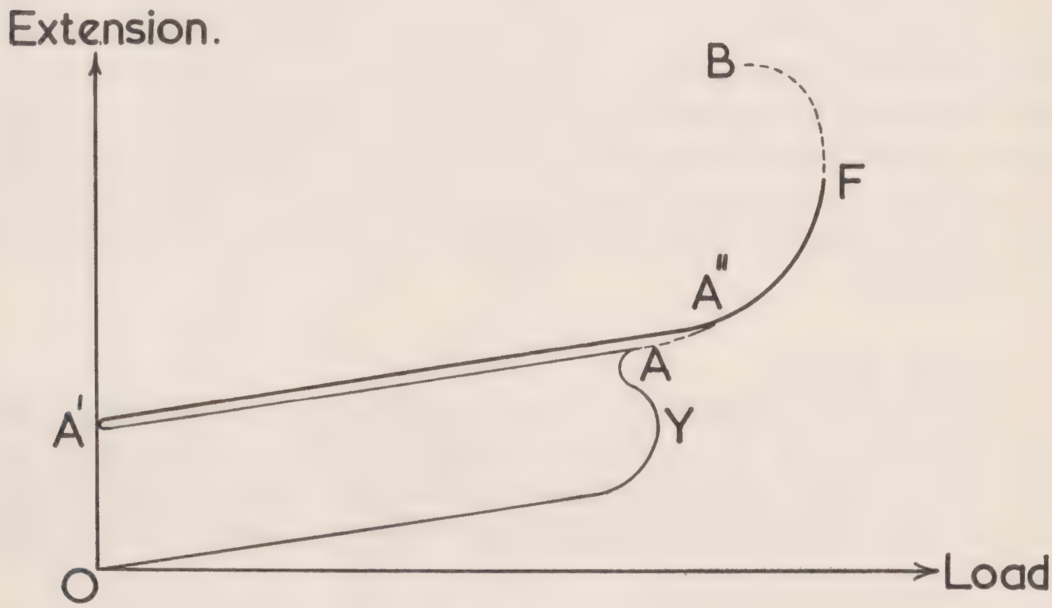


Fig. 8.7

however, to any shape of body. Also, if the body is capable of withstanding a compression instead of a tension, then similar elastic effects will be noticed during the compression. A graph showing the same

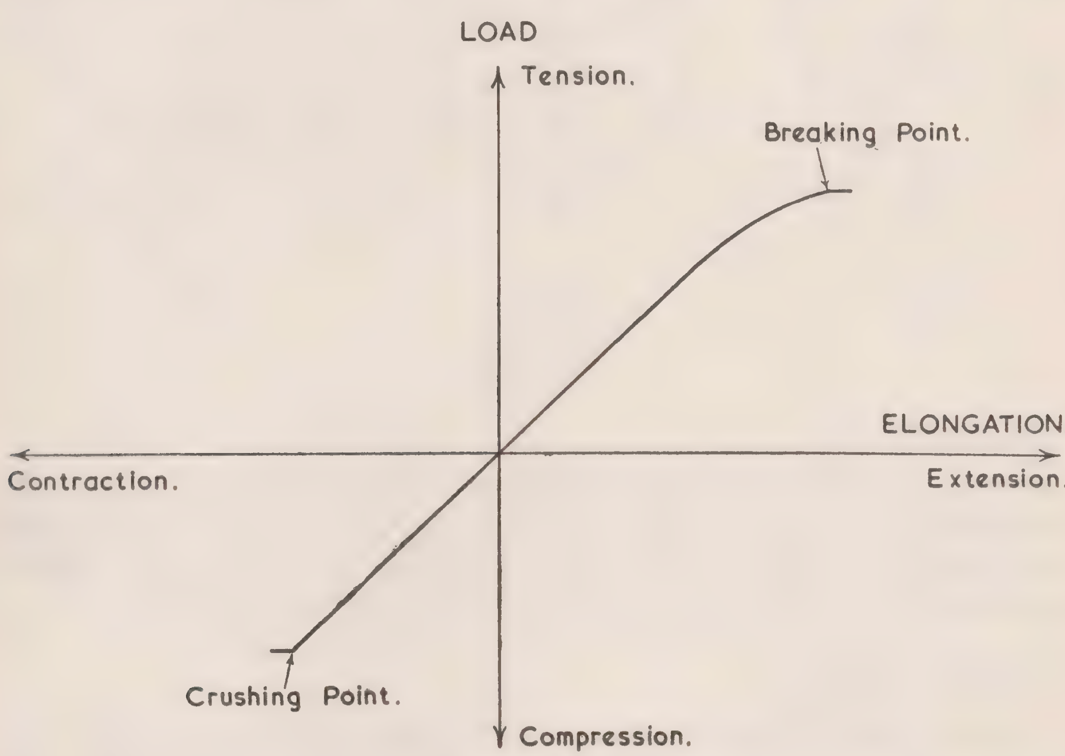


Fig. 8.8

material subjected to both tension and compression may not, however, be symmetrical; the curve for concrete shown in Fig. 8.8. is an example.

8.3 Stress and Strain

The stretching force applied to the wire in the previous section was just a load attached to the wire; the *stress* to which the wire is subjected is defined as the *stretching force per unit area of cross-section of the wire.* The diagrams above have, of course, been plotted in terms of load and not of stress—had stress been used, then the portion F to B of Fig. 8.7. would not have curved backwards, but the falling load would have been offset by the reduction in cross-sectional area, so maintaining the stress constant.

(a) Tensile Stress

A stress may be applied to a block of material in many ways. The stress considered in the previous section is applied in one direction only, this is usually called an *axial* or *tensile stress* (Fig. 8.9 (*a*)). If the material is in a rigid form, the direction of the force may be reversed, giving a compressive stress.

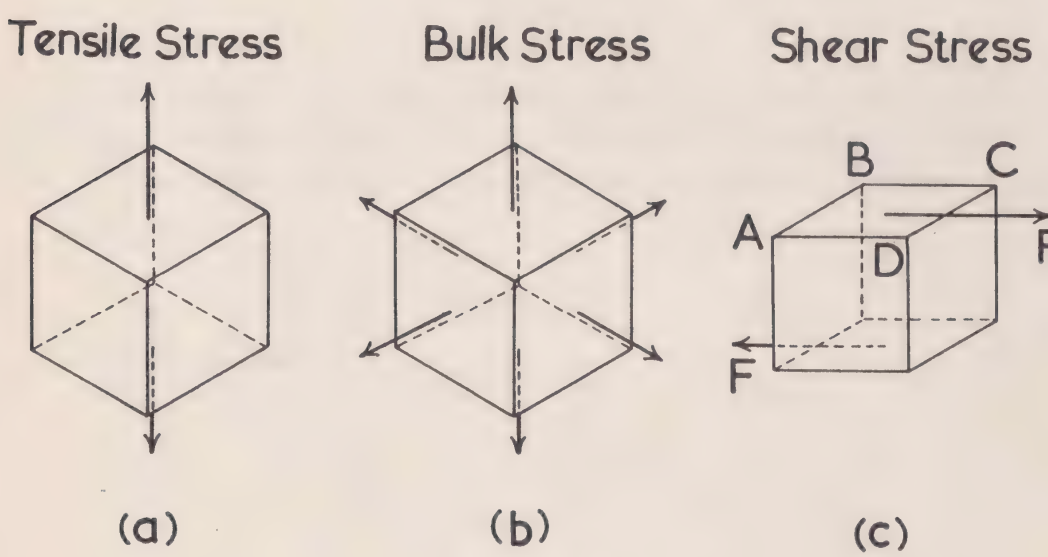


Fig. 8.9

(b) Bulk Stress

A stress may also be applied equally to all surfaces of a body, as for example if the body were immersed in a fluid and subjected to a hydrostatic pressure (Fig. 8.9 (*b*)); this is known as a *bulk stress* and its magnitude is given by the normal force per unit surface area.

(c) Shear Stress

Lastly, a set of forces may be applied tangentially over an opposing pair of surfaces as shown in Fig. 8.9 (*c*); this is called a *shear stress.* The

magnitude of the shear is equal to the shear force F divided by the area of the surface $ABCD$, over which it is applied.

(d) Tensile, Bulk and Shear Strain

Each of the stresses defined above will cause a change in the shape and size of the body; the change produced in the body expressed in a dimensionless form is called the *strain*.

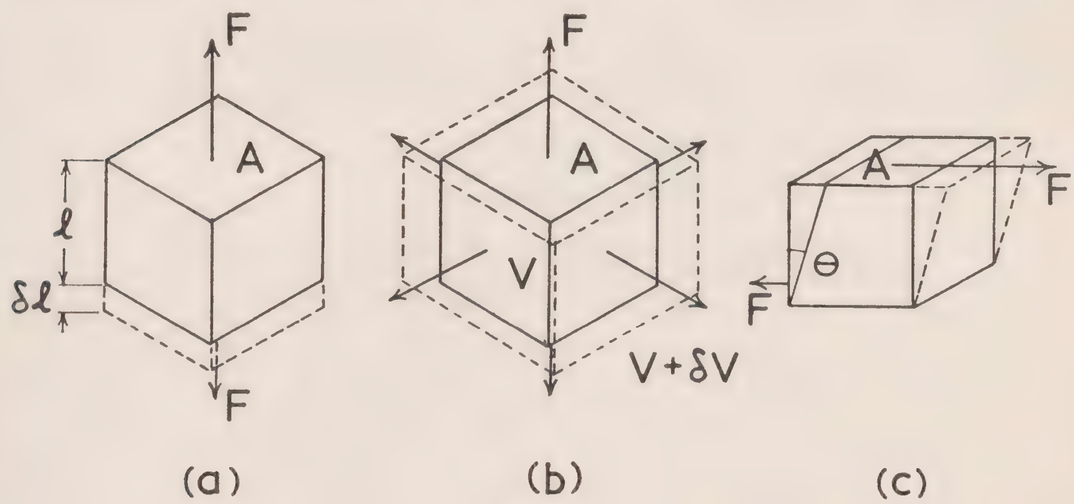


Fig. 8.10

Thus a tensile stress produces a tensile strain given by $\delta l/l$ (Fig. 8.10 (*a*)) while a bulk stress gives a bulk strain of $\delta V/V$ (Fig. 8.10 (*b*)). The shear strain is defined as the angular deformation θ (Fig. 8.10 (*c*)).

Notice that all three stresses have the same dimensions, $\left[\dfrac{\text{Force}}{\text{Area}}\right]$ or $[ML^{-1}T^{-2}]$ and all three strains are dimensionless.

Bulk stress produces a change in size only of the body, whilst a shear stress produces only a change of shape. Each of the three stresses can be applied to a body at the same time. It is found that the resultant strain is the sum of the strains which would have arisen if each of the stresses had been applied individually. By a judicious application of the three stresses it is possible to produce any desired strain, either of size or shape.

8.4 The Elastic Moduli

For any material, certain moduli can be defined which measure the ability of the material to resist the various types of stress to which it may be subjected. In general, any elastic modulus is given by the ratio $\left[\dfrac{\text{stress}}{\text{strain}}\right]$, but since it is possible to subject a body to three types of stress, there will also be three types of modulus, and in the definition

of each one, the appropriate form of stress and resultant strain must be stated.

These moduli have constant values only if the stress is restricted to values lower than the limit of proportionality, i.e. over the Hooke's Law part of the curve, and it is over this range that the definitions given below should be applied.

(a) Young's Modulus

The ability of the material of a body to withstand a longitudinal or tensile stress (Fig. 8.10 (*a*)) is called *Young's Modulus*, symbol Y; thus

$$Y = \frac{\text{Longitudinal stress}}{\text{Longitudinal strain}}$$

$$\text{or } Y = \frac{\text{Longitudinal force} \div \text{cross sectional area}}{\text{extension} \div \text{original length}}$$

$$= \frac{F/A}{\delta l/l}$$

$$= \frac{Fl}{A\delta l} \qquad (1)$$

The value of Young's Modulus is most conveniently found when the material is available in the form of a wire; the wire is subjected to a known stretching force, the extension measured, and Young's Modulus found by substituting the measured values in Equation (1). For experimental details the reader is referred to *Experimental Physics*, by Daish and Fender, published by English Universities Press, Ltd.

(b) Bulk Modulus

If a body is subjected to a bulk stress, then the *Bulk Modulus*, symbol K, for the material of the body is defined by:

$$K = \frac{\text{Bulk stress}}{\text{Bulk strain}}$$

$$K = \frac{\text{Compressive (or tractile) force} \div \text{surface area}}{\text{change in volume} \div \text{original volume}}$$

or, from Fig. 8.10 (*b*):

$$K = \frac{F/A}{\delta V/V}$$

$$= \frac{FV}{A\delta V} \qquad (2)$$

(c) Modulus of Rigidity

Finally, if a shear stress is applied to a body, the *Modulus of Rigidity*, symbol n, of its material is given by:

$$n = \frac{\text{Shear stress}}{\text{Shear strain}}$$

$$= \frac{\text{Shearing force} \div \text{area over which force is applied}}{\text{Angular deformation}}$$

Referring to Fig. 8.10 (c), this gives:

$$n = \frac{F/A}{\theta},$$

$$\text{thus } n = \frac{F}{A\theta} \qquad (3)$$

(θ must be measured in radians.)

The dimensions of stress are $[ML^{-1}T^{-2}]$, while strain is dimensionless, thus all of these elastic moduli have the dimensions $[ML^{-1}T^{-2}]$. The c.g.s. unit for all the elastic moduli is the gm.cm.$^{-1}$sec^{-2} or dyne.cm^{-2} while the M.K.S. unit is the newton.m^{-2}. The f.p.s. unit is the poundal.ft^{-2} and the gravitational unit is the lb-wt per sq ft, but the engineer more often uses the ton-wt per sq in.

(d) Poisson's Ratio

It is found by experiment that, when a body is subjected to a longitudinal stress, a lateral contraction takes place; the ratio of the two

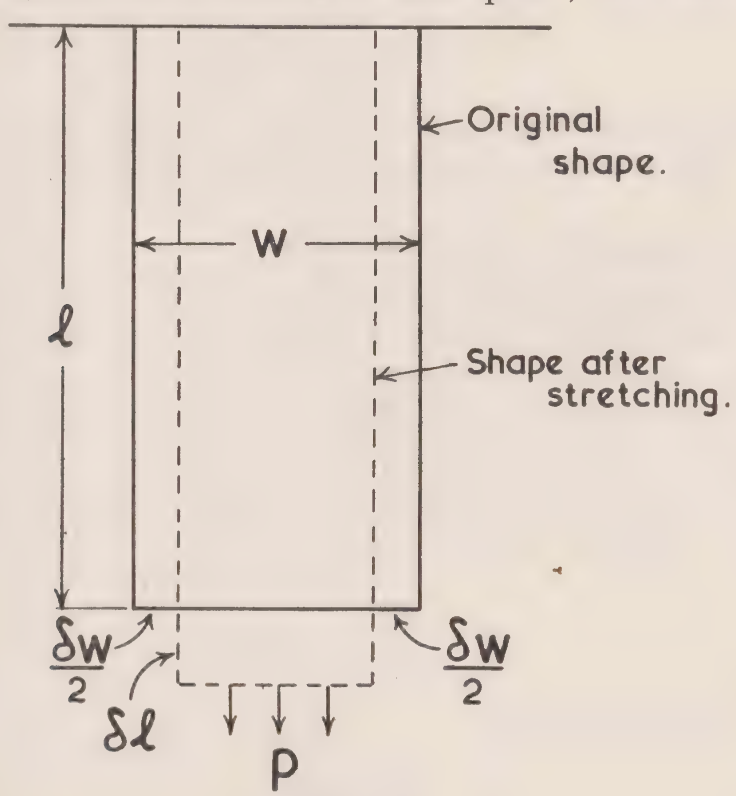


Fig. 8.11

dimensional changes, expressed as fractions of the original dimensions, i.e. the ratio of the strains, is called *Poisson's Ratio*, symbol δ; thus, from Fig. 8.11:

$$\delta = \frac{\delta w/w}{\delta l/l}$$

$$= \frac{l\delta w}{w\delta l} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where l is the original length of the body, w its original width, δl the extension and δw the contraction in width.

Poisson's ratio is dimensionless and thus a pure number.

8.5 Relation between the Elastic Constants

Any one of the strains discussed above can be produced in a body by applying to it a combination of the other two stresses; for example, a bulk strain can be produced by applying three longitudinal stresses mutually at right angles to each other.

By this method it is possible to calculate various relations between the elastic constants; the work is not easy and here the results only are stated.

Thus: $$Y = 3K\,(1 - 2\delta) \quad . \quad . \quad . \quad (5)$$

$$Y = 2n\,(1 + \delta) \quad . \quad . \quad . \quad (6)$$

$$\frac{1}{Y} = \frac{1}{3n} + \frac{1}{9K} \quad . \quad . \quad . \quad . \quad (7)$$

$$K = \frac{2n\,(1 + \delta)}{3\,(1 - 2\delta)} \quad . \quad . \quad . \quad (8)$$

It will be noticed that each equation contains three of the four constants; thus if two are known the others can be calculated.

8.6 Work done in Stretching a Wire

If a wire of normal length l is stretched by an amount δl, the force producing this extension moves through the distance δl and consequently does some work. Unfortunately the work cannot be calculated merely by multiplying together the force and the distance it moves, since the force is not constant but increases as the wire is stretched.

In this case, the work done may be calculated as follows. If a wire of length l is stretched by a force F, then the extension δl is given by the equation:

$$Y = \frac{Fl}{A\delta l}, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

which can be rearranged as:

$$F = \frac{YA}{l} \cdot \delta l \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Equation (10) indicates that the stretching force is proportional to the extension, since the factor YA/l is a constant; thus as the extension grows from zero to δl, the stretching force will increase linearly from zero to its value F. Because of this linear growth, the work can be calculated by multiplying the extension by the *average* force, or:

$$W = \delta l \cdot \frac{F}{2} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

and substituting for F from Equation (10) gives:

$$W = \frac{1}{2} \cdot \frac{YA}{l} \cdot (\delta l)^2 \quad . \quad . \quad . \quad . \qquad (12)$$

Now the work done in stretching a wire is stored as potential (or strain) energy in the wire and can be recovered as useful work when the wire relaxes; thus:

$$\text{Potential energy of stretched wire} = \frac{1}{2} \cdot \frac{YA}{l} \cdot (\delta l)^2$$

But the volume of the wire is Al, therefore:

$$\text{Potential energy per unit volume} = \frac{1}{2} \cdot \frac{YA}{l} \cdot \frac{(\delta l)^2}{Al}$$

$$= \frac{1}{2} Y \left(\frac{\delta l}{l}\right)^2 \quad . \quad . \qquad (13)$$

$$= \frac{1}{2} Y \,(\text{Strain})^2 \quad . \qquad (14)$$

Alternatively, Equation (13) may be rearranged as:

$$\text{Potential energy per unit volume} = \frac{1}{2} \cdot \frac{Y\delta l}{l} \cdot \frac{\delta l}{l}.$$

From Equation (10) it will be seen that $Y\delta l/l$ is equal to F/A, which in turn is equal to the stress applied to the wire; thus the equation above may be written as:

$$\text{Potential energy per unit volume} = \frac{1}{2}\,(\text{Stress}).(\text{Strain}). \qquad (15)$$

This case may also be treated graphically. The stretching force is plotted against the extension as in Fig. 8.12; an increase in the force from f to $f + \delta f$ increases the extension from ε to $\varepsilon + \delta\varepsilon$. The work done by this force is $(f + \delta f/2)\,.\,\delta\varepsilon$, which is the area of the strip shaded in the diagram. If, then, a force AB of magnitude F produces an overall extension to A (equal to δl), the total work will be the sum of all such strips. This is the area enclosed by the curve; it is equal to $\delta l . F/2$ as before (see Equation 11).

If the extension and load are not linearly related and the graph is a curve, the work done is still given by the area under the curve. If the material shows a different behaviour during stretching and

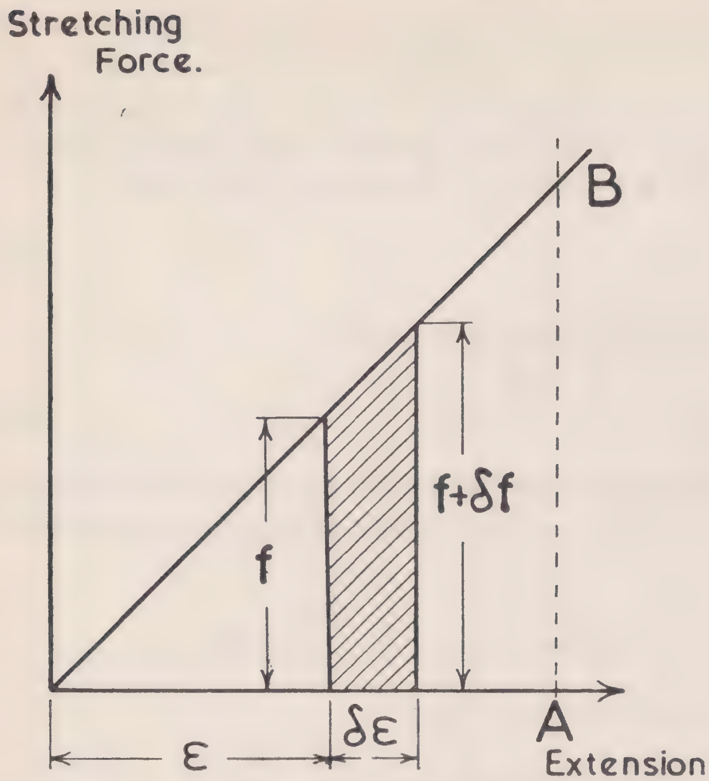


Fig. 8.12

relaxing, as shown in Fig. 8.13, the work done on the material while stretching it is represented by the area *ODCBA*, whilst on relaxing, the material only gives up work represented by the area *OD′C′BA*. An amount of work represented by the area *ODCBC′D′* is thus lost during each cycle.

8.7 Bending of Beams

If the value of Young's Modulus for the material of a railway track, for example, has to be found, it is not permissible to measure this constant for the same material when formed into a wire, because the elastic properties of a material are modified considerably as it is drawn into a wire (see page 206). In cases such as this, when the material is available as a stout rod, beam or girder, Young's Modulus can be measured by taking measurements on the couples needed to bend the beam. (The term 'beam' is used here to indicate any form of the material sufficiently stout to be rigid and not flexible like a wire.)

If a beam is bent into the shape shown in Fig. 8.14, then the layers of the beam on

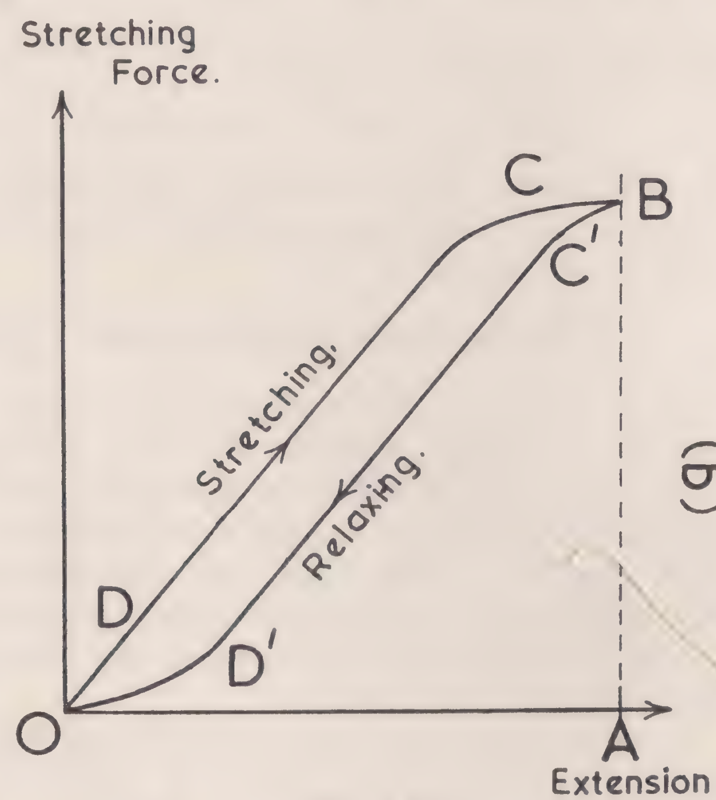


Fig. 8.13

the outside of the curve will be stretched and in a state of tension, while the layers on the inside of the curve will be compressed. Somewhere in the middle of the beam is a layer which, in the bent

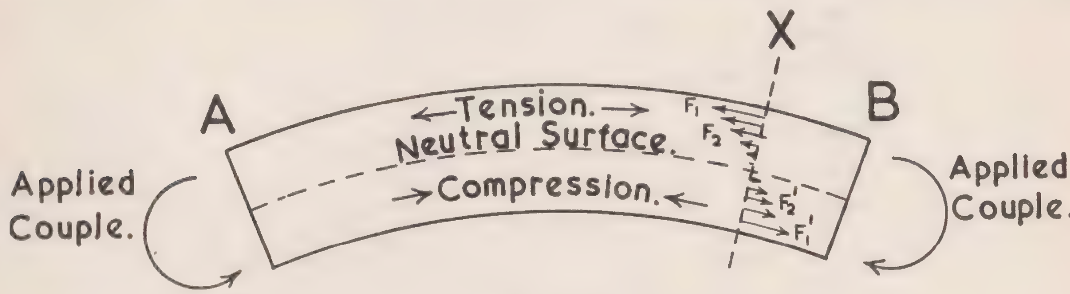


Fig. 8.14

condition, retains its original length; this is called the *Neutral Surface*, and is neither in a state of tension nor compression.

Moving outwards in either direction from the neutral surface, the layers become more and more stretched or compressed, and consequently the forces of tension or compression in the layers increase also.

Consider a cross-section of the beam at X. The tension in the various layers of the portion AX of the beam exert forces F_1, F_2, F_1', F_2', etc. on the portion XB. These forces, taken in pairs F_1F_1', F_2F_2', etc., constitute couples, all of which try to rotate the portion XB in an anticlockwise direction. The effect of all these couples can be added up into one couple, called the *Internal Bending Moment*, and, using a mathematical treatment that is rather beyond the standard of this book, it is possible to show that if the beam is bent into an arc of a circle of radius R, then the internal bending moment is given by:

$$\text{Internal bending moment} = \frac{YAk^2}{R} \qquad . \qquad . \qquad (16)$$

where Y is Young's Modulus for the material of the beam, A is its area of cross-section and k is the radius of gyration of the cross-sectional area of the beam about the neutral surface as axis.

The factor (YAk^2) is called the *Flexural Rigidity* of the beam.

The student may wonder why the radius of gyration should appear in this expression. The answer is very simple—if the mathematics of the problem were followed through carefully, an integral appears which is identical with one normally encountered when working out moments of inertia. The moments of inertia and radii of gyration of all common cross-sections are listed in books of tables, hence we can avoid the labour of evaluating the integral by reference to these tables.

Referring once more to Fig. 8.14, it is evident that the part XB of the beam can be in equilibrium only if the total couple applied to it is zero. The couples acting on it are the internal bending moment and the external couple holding the beam in the bent shape, and if the resultant couple is to be zero, these must obviously be equal and opposite.

The application of this principle may be illustrated by considering a beam clamped horizontally at one end and loaded with a weight W at the other, as in Fig. 8.15. If an element of the beam at a distance x

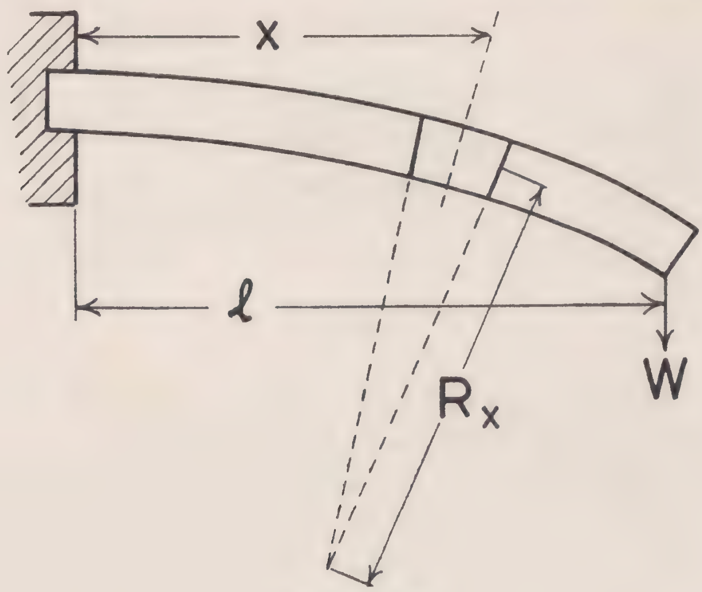


Fig. 8.15

from the support is bent into an arc of radius R_x, then the internal bending moment at that point is YAk^2/R_x. If the weight of the beam itself is ignored, then the couple bending the element of the beam is equal to the moment of the weight W, i.e. $W(l - x)$, and if the beam is in equilibrium, then:

$$\frac{YAk^2}{R_x} = W(l - x)$$

$$\text{or } R_x = \frac{YAk^2}{W(l - x)} \quad . \quad . \quad . \quad . \quad (17)$$

This equation shows that each element of the beam is bent into an arc of different radius, depending on the position of the element along the beam, i.e. on the value of x; the final shape of the beam is rather complex, but nevertheless can be calculated from Equation (17), using more advanced mathematics.

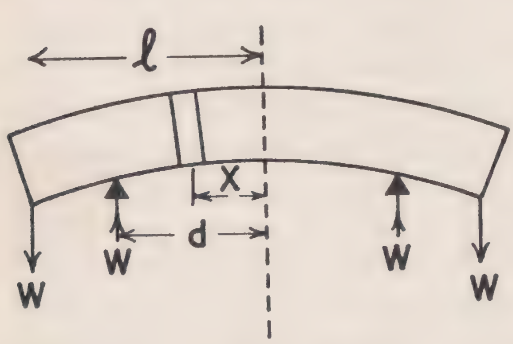


Fig. 8.16

A simpler case arises if the beam is supported as shown in Fig. 8.16 and loaded with weights W at each end. There will be a reaction W at each of the supports, and thus the couples bending the element of the beam at a distance x from the mid-point are $W(l - x)$ anticlockwise, due to the load and $W(d - x)$ clockwise due to the reaction at the support.

Thus the total couple in the anticlockwise sense is given by:

$$\Gamma = W(l - x) - W(d - x)$$
$$= W(l - d),$$

provided that the element lies between the two supports.

Equating this to the internal bending moment gives:

$$\frac{YAk^2}{R_x} = W(l - d)$$

$$\text{or } R_x = \frac{YAk^2}{W(l - d)} \qquad . \quad . \quad . \quad . \quad . \quad (18)$$

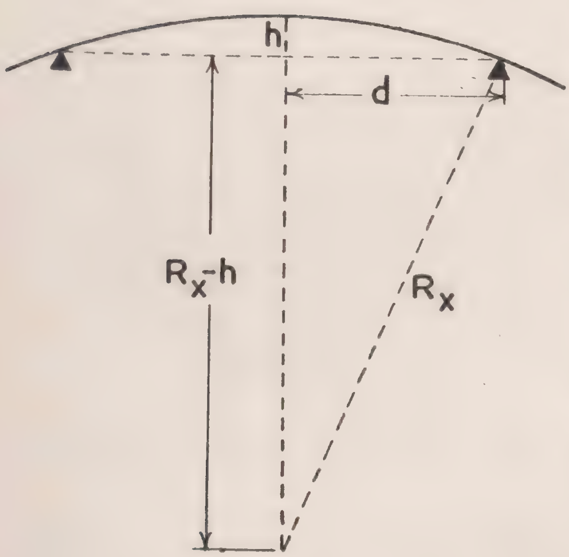


Fig. 8.17

Now the right-hand side of this equation is a constant, thus R_x is a constant, i.e. all elements of the beam between the supports are bent into arcs of the same circle; between the supports the beam itself is therefore a circular arc of radius R_x given by Equation (18).

If the central point of the beam rises a distance h when the loads are added, then from Fig. 8.17:

$$R_x^2 = d^2 + (R_x - h)^2,$$
$$\text{thus } R_x = (d^2 + h^2)/2h$$

and substituting this value in Equation (18) gives:

$$Y = \frac{W(d^2 + h^2)\ (l - d)}{2hAk^2} \qquad . \quad . \quad . \quad (19)$$

This provides a useful laboratory method of measuring Y for the material of a beam; a brass rod, for example, of rectangular section 3×1 cm and 120 cm long, supported 20 cm from each end, will rise by about a millimeter at the centre when a load of 2 kg is hung at each end. This distance can be measured very accurately with a travelling microscope and a good value for Y obtained.

8.8 Torsion of a Tube or Cylinder

The methods normally used for measuring the modulus of rigidity of a material depend on determining the couple needed to twist a wire through a known angle. The couple and the modulus of rigidity can be related as follows.

Consider a section of thin-walled tube of radius r and length l (Fig. 8.18) fixed at its lower end and twisted by a couple at its upper end;

then a radius OA drawn on the upper surface will turn through a small angle θ to the position OB, and a line AC, drawn parallel to the axis on the surface of the tube, will be inclined until it takes up a position CB, turning through an angle φ.

The angles θ and φ can be related, since AB is an arc of a circle centre O, thus arc $AB = r\theta$ where r is the radius of the tube; also if φ is small,

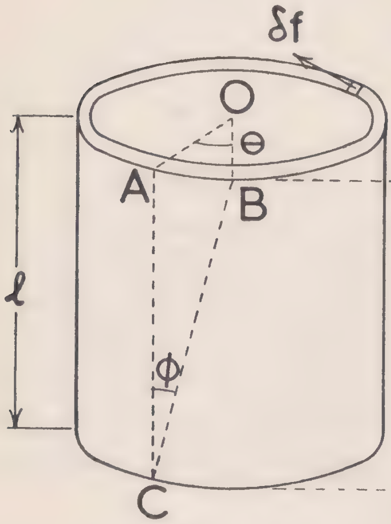


Fig. 8.18

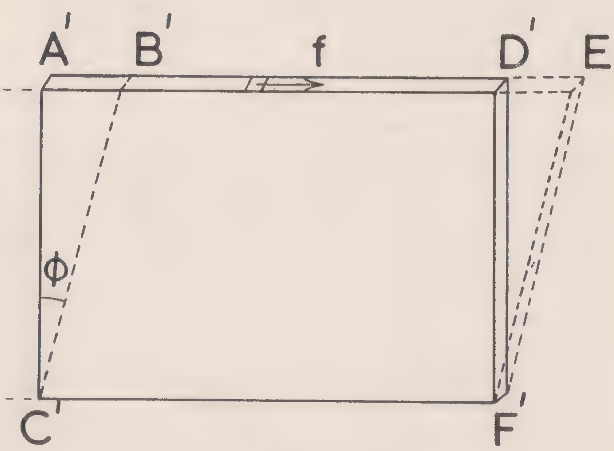


Fig. 8.19

AB can be considered as the arc of a circle centre C, thus arc $AB = l\varphi$ and:

$$r\theta = l\varphi \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Now the tube can be considered as a flat plate $A'D'F'C'$ wrapped into a cylinder (Fig. 8.19); as the tube is twisted, the plate is sheared into the shape $B'E'F'C'$, the distortion of the material is the same in each case, hence the stress needed to twist the cylinder can be calculated from the stress needed to shear the plate.

Suppose that a force f shears the plate from the position $A'D'F'C$ to the position $B'E'F'C'$. $B'E'$ is equal to $2\pi r$ and if the thickness of the wall of the cylinder is δr, the area over which the force is applied is $2\pi r\delta r$, thus the shearing stress is $f/2\pi r\delta r$ and the modulus of rigidity of the material of the tube is given by:

$$n = \frac{f/2\pi r\delta r}{\varphi}$$

$$= \frac{f}{2\pi\varphi r\delta r}.$$

Substituting θ for φ from Equation (20) gives:

$$n = \frac{fl}{2\pi\theta r^2\delta r}$$

and rearranging this leads to:

$$f = \frac{2\pi n\theta r^2 \delta r}{l} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Now, when the plate is shaped into a circular cylinder, the force f will be made up of a number of elements δf, tangential to the wall of the tube; each of these will have a moment $r\delta f$ about the axis of the tube. The total moment of all of the elements will be $r\Sigma\delta f$ which is equal to rf. This is the couple Γ twisting the tube, thus:

$$\Gamma = rf,$$

and substituting for f from Equation (21) gives:

$$\Gamma = \frac{2\pi n\theta r^3 \delta r}{l} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

This equation is applicable only to a tube with very thin walls, but it can be extended for a thick-walled tube or a solid rod as follows. A rod can be considered to be made up of a number of cylindrical tubes of different radii inside each other and all twisted through the same angle at the end. In this case the couple calculated in Equation (22) is not the couple required to twist the rod but only the element of couple required to twist a cylindrical element of the rod; it should thus be written as $\delta\Gamma$ where:

$$\delta\Gamma = \frac{2\pi n\theta r^3 \delta r}{l}$$

Hence the total couple Γ needed to twist a rod of radius a is the sum of all the elements of couple needed to twist the cylindrical elements. These have radii varying from zero to a, or:

$$\begin{aligned} \Gamma &= \frac{2\pi n\theta}{l}\int_0^a r^3 dr \\ &= \frac{2\pi n\theta}{l}\left[\frac{r^4}{4}\right]_0^a \\ &= \frac{\pi n\theta a^4}{2l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23) \end{aligned}$$

If this method is applied to a thick-walled tube, with outside and inside radii a, b respectively, then:

$$\begin{aligned} \Gamma &= \frac{2\pi n\theta}{l}\int_b^a r^3 dr \\ &= \frac{\pi n\theta}{2l}(a^4 - b^4) \quad . \quad . \quad . \quad . \quad . \quad (24) \end{aligned}$$

It will be noticed that both of these expressions are of the form

$$\Gamma = c\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where c is a constant depending on the dimensions and material of the rod or tube; c is called the *Torsional Rigidity* of the body thus:

$$\text{Torsional rigidity} = \frac{\pi n a^4}{2l} \text{ for a solid cylinder,}$$

$$\text{and Torsional rigidity} = \frac{\pi n(a^4 - b^4)}{2l} \text{ for a hollow cylinder.}$$

It will be noticed from Equation (25) that the torsional rigidity is numerically equal to the torque required to twist the rod through one radian—it is thus a measure of the torsional strength of the rod. This equation also indicates that the torque needed to deform a wire is proportional to the angle through which it is twisted. This fact is true for most shapes of bodies and also holds for twists of several complete turns, provided that the axis about which the body is twisted is long compared with all its other dimensions.

8.9 Methods of Measuring the Modulus of Rigidity

(a) Barton's Static Method

The twisting of a cylindrical rod, using the apparatus shown in Fig. 8.20, provides a convenient method for measuring the modulus of rigidity.

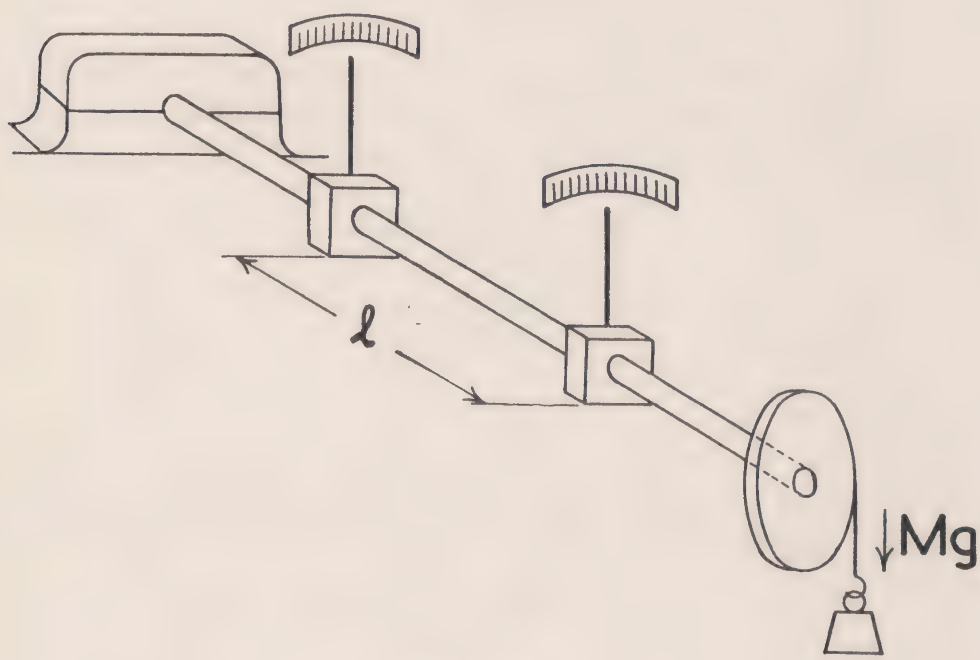


Fig. 8.20

A pulley is attached to the end of the rod, suitably mounted in bearings, and a weight is hung on a tape wound round the pulley. The twist in a measured length of the rod is read off angular scales by pointers

attached to the rod. If the radius of the pulley is R, then the couple applied to the rod is MgR, and substituting this in Equation (23) gives:

$$MgR = \frac{\pi n \theta a^4}{2l}$$

$$\text{or } n = \frac{2lMgR}{\pi \theta a^4} \quad . \quad . \quad . \quad . \quad (26)$$

This assumes that θ is measured in radians; if, however, it is measured in degrees, then:

$$n = \frac{360lMgR}{\pi^2 \theta a^4} \quad . \quad . \quad . \quad . \quad (27)$$

(b) Torsional Pendulum

If the material is available in the form of a wire instead of a rod, its modulus of rigidity may be found by measuring the period of a bob executing torsional vibrations when suspended by the wire. The apparatus is shown in Fig. 8.21.

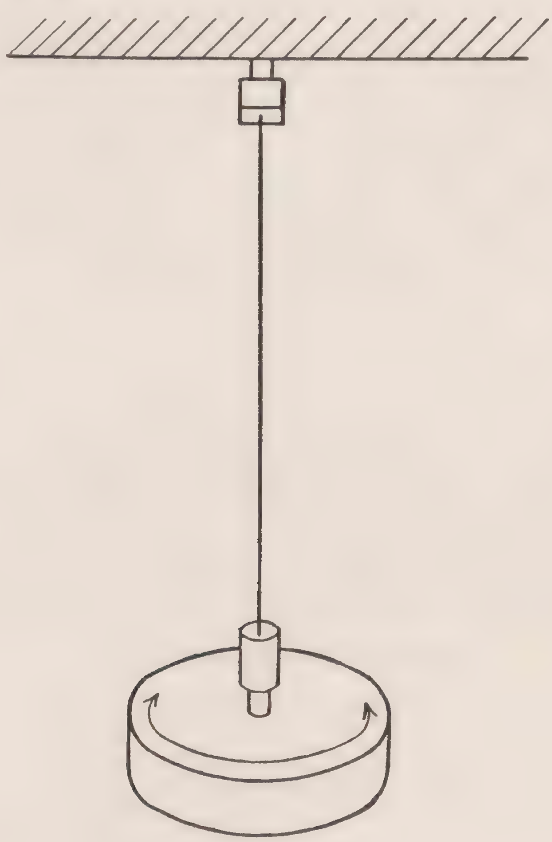


Fig. 8.21

If the bob is twisted through an angle θ then the restoring couple developed by the wire is given by Equation (25), i.e.:

$$\Gamma = c\theta,$$

where c is the torsional rigidity of the wire.

This couple is proportional to the angular displacement and tends to restore the bob to its original position; thus the angular motion of the bob is simple harmonic (see page 104), and its period is given by:

$$\tau = 2\pi\sqrt{\frac{\text{Moment of inertia of bob}}{\text{Restoring couple at unit angular displacement}}}.$$

Now, when θ is equal to unity, Γ is numerically equal to c, thus:

$$\tau = 2\pi\sqrt{\frac{I}{c}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where I is the moment of inertia of the bob. (This can normally be calculated from its mass and dimensions.)

If the wire is a solid circular cylinder, then c is equal to $\pi na^4/2l$, and substituting this in Equation (28) gives:

$$\tau = 2\pi\sqrt{\frac{2lI}{\pi na^4}}$$

$$\text{or } n = \frac{8\pi lI}{\tau^2\, a^4} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

from which the modulus of rigidity can be calculated.

It should be noted that in this work the approximation 'sin $\theta \simeq \theta$ for small values of θ', which is so common in problems on simple harmonic motion, has not been used. No such restriction is therefore placed on the motion of the torsional pendulum and if necessary the bob may rotate through a large angle without violating the theory.

8.10 Elasticity of a Gas

A gas subjected to a bulk stress, i.e. a pressure, changes its volume and thus undergoes a bulk strain. Hence a bulk modulus can be calculated for a gas just as for a solid.

Consider a mass of gas which at pressure p occupies a volume V. If the pressure increases by δp and the volume decreases by δV, then:

$$\text{Increment in stress} = \delta p,$$

$$\text{Increment in strain} = \delta V/V,$$

$$\text{thus bulk modulus} = \frac{\delta p}{\delta V/V}$$

$$= \frac{V\delta p}{\delta V} \quad . \quad . \quad . \quad (30)$$

If the gas obeys Boyle's Law and the change takes place isothermally, then:

$$(p + \delta p)\,(V - \delta V) = pV$$

$$\text{or } - p\delta V + V\delta p = 0.$$

[The product $(\delta p \,.\, \delta V)$ can be ignored as δp and δV tend to zero.]

$$\text{Thus } \frac{V\delta p}{\delta V} = p,$$

and substituting this value in Equation (30) leads to:

$$\text{Bulk modulus} = p \quad . \quad . \quad . \quad . \quad (31)$$

or the bulk modulus of a gas under isothermal conditions is equal to its pressure (which must, of course, be expressed in absolute units—dyne.cm^{-2}). This is normally called the *Isothermal Elasticity* of the gas, and it will be noticed that it is not a constant. A gas can be likened to a solid which gets stronger the more it is compressed—equal increments in pressure producing smaller and smaller changes in volume. An ordinary solid, provided the elastic limit is not exceeded, shows equal changes in volume for equal increments in pressure.

If the change in the gas takes place adiabatically, then the gas obeys the law

$$pV^{\gamma} = \text{constant}$$

$$\text{thus } (p + \delta p)\,(V - \delta V)^{\gamma} = pV^{\gamma}$$

$$\text{or } \left(1 + \frac{\delta p}{p}\right)\left(1 - \frac{\delta V}{V}\right)^{\gamma} = 1.$$

If $\delta V/V$ is small, the term $(1 - \delta V/V)^{\gamma}$ can be expanded by the binomial theorem; ignoring powers of $\delta V/V$ higher than the first, this gives:

$$\left(1 + \frac{\delta p}{p}\right)\left(1 - \frac{\gamma\delta V}{V}\right) = 1$$

$$\text{or } \frac{\delta p}{p} - \frac{\gamma \delta V}{V} = 0$$

if the product of small terms is ignored.

Rearranging this equation gives

$$\frac{V\delta p}{\delta V} = \gamma p$$

and substituting in Equation (30) leads to:

$$\text{Adiabatic bulk modulus} = \gamma p \quad . \quad . \quad . \quad (32)$$

This is also called the *Adiabatic Elasticity* of the gas.

Summary of New Quantities Introduced in this Chapter

Quantity	*c.g.s. unit*	*f.p.s. unit*	*Gravitational units*
Stress . . .	dyne.cm^{-2}	poundal.ft^{-2}	gm-wt.cm^{-2} lb-wt.ft^{-2} lb-wt per sq in.
Strain . . .	Pure Number		
Young's Modulus Bulk Modulus Modulus of Rigidity	dyne.cm^{-2}	poundal.ft^{-2}	kg-wt.cm^{-2} tons-wt per sq in.
Poisson's Ratio .	Pure Number		

EXERCISES 8

1. What is Hooke's Law?

   Two long light springs have the same unstretched length. A certain weight hung on the end of one spring produces an extension of one inch; the same weight extends the other spring by two inches. What will be the extension produced by this weight when (*a*) the springs are joined at their ends and hang side by side, and (*b*) when one end of one is joined to one end of the other so that they hang in the same straight line, and the weight is attached to the lower end of the combination? (Oxford Univ. Schol.)

2. Draw a labelled diagram of the apparatus you would use to determine Young's modulus for the material of a wire. Describe in detail how you would take all the readings necessary for an accurate result. Give reasonable values for the following quantities: (*a*) the length of the wire, (*b*) its diameter, (*c*) the largest weight you would expect to put on, (*d*) the maximum extension you would expect to obtain.

   A vertical steel wire and a parallel brass wire, each 2 metres long and 0·2 mm. in diameter, hang from the ceiling and are 50 cm. apart. The lower ends are attached to points 50 cm. apart on a light horizontal bar. What weight must be hung from the bar to extend both wires by $\frac{1}{2}$ cm. and at what distance from the steel wire must it be attached?

   (Young's moduli: steel 2×10^{12} dyne-cm.$^{-2}$; brass 10^{12} dyne-cm.$^{-2}$.)

   (Cambridge G.C.E. Advanced level.)

3. Define the terms *stress* and *strain* and *Young's modulus*. State Hooke's law and describe the method you would use to verify it for a wire.

   A brass wire AB, 200 cm. long, is supported at A and is kept taut in a vertical position by means of a platform at B, on which additional 'weights' may be added. From the under side of the platform a steel wire, CD, also 200 cm. long, is supported at C and is kept taut by a similar platform at D. Both wires are 0·7 mm. in diameter.

By how much will the end D be depressed when a load of 4 kgm. is placed (a) at B and (b) at D?

(Young's modulus for steel $= 2{\cdot}2 \times 10^{12}$ dynes/cm.^{2} and for brass $= 1{\cdot}1 \times 10^{12}$ dynes/cm.^{2})

(London Univ. G.C.E. Advanced level.)

4. Define Young's modulus of elasticity and describe how you would measure it for a metal in the form of a long wire.

A rigid horizontal bar, of uniform cross-section 100 cm. long and mass 360 gm., is supported by two uniform vertical wires of steel and copper respectively. Each wire is 200 cm. long and 0·036 cm.^{2} in cross-section: the copper wire is attached to one end of the bar and the steel wire at such a distance x from this end that both wires suffer the same extension. Calculate (i) the distance x, (ii) the tension in each wire and (iii) the extension of each wire.

(Young's modulus for copper $= 10 \times 10^{11}$ dynes/cm.^{2}
Young's modulus for steel $= 20 \times 10^{11}$ dynes/cm.^{2})

(London Univ. G.C.E. Advanced level.)

5. Define *stress, strain, Young's modulus.* Give a brief account of an experimental method for determining the value of Young's modulus for the material of a wire.

A weight of 20 kgm. hangs by a support 5 metres long compounded of two wires, respectively of brass and steel, each 5 metres long, joined together at both ends. If the cross-sectional area of each wire is 0·01 sq.cm., by how much will the wires stretch when the weight is applied?

(Young's modulus for steel $= 20 \times 10^{11}$ C.G.S. units; for brass $= 10 \times 10^{11}$ C.G.S. units.) (Cambridge H.S.C.)

6. Explain the terms *elastic limit, yield point* and *modulus of elasticity.* Describe how you would determine Young's modulus for a material in the form of a wire.

Young's modulus for steel is 2×10^{12} dyne cm.$^{-2}$ Find the force required to extend by 5 mm. a steel wire 3 metres long of diameter 2 mm., and also the work done in producing this extension.

(Oxford H.S.C.)

7. Define the terms *stress, strain, modulus of elasticity,* and describe how you would measure Young's modulus for a material in the form of a wire.

The breaking stress for steel is $1{\cdot}5 \times 10^{10}$ dyne cm.$^{-2}$, and Young's modulus is 2×10^{12} dyne cm.$^{-2}$ Assuming that there is no change in volume, that the wire thins uniformly throughout its length, and that Hooke's law holds for the whole of the extension, calculate the percentage change in diameter of a steel wire which is stretched until it breaks. (Oxford G.C.E. Advanced level.)

8. Define *Young's modulus of elasticity.* Derive an expression, in terms of the strain and elastic modulus, for the energy stored per c.c. in a strained body of isotropic material when it is subjected to a pull in one direction.

If the breakdown strength of steel is 20 kgm. per sq.mm., calculate

the maximum amount of energy per c.c. which can be stored in the metal when stretched.

(Young's modulus of steel $= 2 \times 10^{12}$ dynes/cm.^{2})

(London Univ. G.C.E. Advanced level.)

9. Find the work done in compressing a spring which obeys Hooke's Law.

A railway truck, which with its contents has a mass of 12 tons, when running at 6 m.p.h. strikes an empty truck of mass 2 tons which is at rest against a buffer in a fixed frame. The buffer is in the form of a spring whose natural length is 4 ft. The two trucks are brought instantaneously to rest when the spring is compressed to 2 ft. Find, in tons weight, the force which would hold the buffer compressed by 1 ft. (London Univ. Inter. B.Sc.)

10. A rod has a circular cross-section of area 1 cm.^{2}, a Young's modulus of $3{\cdot}2 \times 10^{11}$ dyne cm.$^{-2}$, and a Poisson's ratio of 0·32. Find the change in its area of cross-section when a load of 100 kgm. is hung from one end. (Take $g = 1000$ cm.sec.$^{-2}$.)

(Cambridge Univ. Schol., King's College Group (Part).)

11. State Hooke's Law, and describe briefly departures from it that occur in practice.

A light elastic string is stretched horizontally between two pegs a distance l apart. When a certain weight is attached to the centre of the string the latter is depressed a distance d ($\ll l$) below the line joining the pegs. What extension would be produced by hanging the same weight on the end of a piece of the string having an unstretched length l? (Oxford Univ. Schol.)

12. Describe experiments you would make to test Hooke's law.

The tension in an elastic material is equal to a constant λ multiplied by the fractional increase in length. A uniform string of this material has mass μ and unstretched length l_0. Calculate the length when a mass M is hung from one end. (Oxford Univ. Schol.)

13. State Hooke's law as applied to a solid subjected to a uniform longitudinal stress, and define the terms *Young's modulus* and *Poisson's ratio.*

A uniform steel wire of unstrained length 1000 metres is fixed at one end and hangs freely under gravity. Determine the length of the wire in this position, given:

Young's modulus for steel $= 20{\cdot}0 \times 10^{11}$ dyne.cm.$^{-2}$; Density of steel $= 7{\cdot}8$ gm.cm.$^{-3}$; Acceleration of gravity $= 981$ cm.sec.$^{-2}$.

(Cambridge Univ. Schol., King's College Group.)

14. Describe the phenomena observed when a wire is strained up to its breaking point. What happens if the load is gradually reduced just before the breaking point is reached?

A circular iron bar is heated to 250° C. and its ends are clamped. What will be the tension in the bar when it has cooled to 15° C.? Young's modulus for iron is 2×10^{12} dynes.cm.$^{-2}$, the coefficient of linear expansion is $1{\cdot}1 \times 10^{-5}$ °C.$^{-1}$ and the diameter of the bar is 1 cm. (Manchester Univ. Schol.)

15. Define *stress, strain, modulus of elasticity.*

Describe how you would measure Young's modulus for a material in the form of a long uniform wire.

An iron bar, 3 sq.cm. in cross-section and 20 cm. long at 0° C., is heated up to 300° C., clamped firmly at the ends, and then cooled down to 0° C. Find the force exerted on the clamps, and the energy stored in the bar.

(Take Young's modulus to be 2×10^{12} dynes per sq.cm., and the coefficient of linear expansion to be 0·000012 per C. degree.)

(Oxford G.C.E. Advanced level.)

16. Sketch a stress-strain diagram for a typical metal in tension and point out its salient features.

A hollow copper tube is 100 cm. long and has internal and external diameters of 1·0 and 1·2 cm.

Find (*a*) the greatest tension to which it can be subjected safely; (*b*) a rough value for the greatest load that may be safely hung on one end when the tube is horizontal and fixed rigidly at the other end; and (*c*) the greatest internal hydrostatic pressure to which the tube may be safely subjected when the external hydrostatic pressure is 10^6 dyne.cm.$^{-2}$.

(Maximum safe tensile stress for copper $= 2 \times 10^9$ dyne.cm.$^{-2}$.)

(Cambridge Univ. Schol., King's College Group.)

17. Describe an experiment to measure the modulus of rigidity of a metal, and give the theory of the method.

A steel rod, 5 mm. in diameter and 120 cm. long, is clamped firmly at one end, and the other end is twisted through an angle of 12°. Find the moment of the couple applied, and the work done in producing this torsion.

(The modulus of rigidity of steel is 8×10^{11} dyne cm^{-2}).

(Oxford H.S.C.)

18. Explain what is meant by a modulus of rigidity, and describe how you would determine the modulus of rigidity of a material provided in the form of a wire.

A vertical wire is fixed at its upper end, and at the lower end carries a uniform cylinder, which is attached to the wire at the middle of its axis, which is horizontal. The cylinder takes 44 sec. to make seven complete torsional oscillations. Calculate the density of the material of the cylinder, using the following information: Length of wire, 40·0 cm.; diameter of wire, 0·1 cm.; modulus of rigidity of the material of the wire, 3×10^{11} dyne.cm.$^{-2}$; diameter of cylinder, 4·0 cm.; length of cylinder, 20·0 cm. (Oxford G.C.E. Schol. level.)

19. Derive an expression for the moment of inertia of a uniform circular disc of mass M, radius r, about a central axis perpendicular to its plane. How would you determine this moment of inertia experimentally?

A circular disc of mass 800 gm., radius 10 cm., is suspended by a wire through its centre perpendicular to its plane and makes 50 torsional oscillations in 59·8 sec. When an annulus is placed symmetric-

ally on the disc, the system makes 50 oscillations in 66·4 sec. Calculate the moment of inertia of the annulus about its axis of rotation.

(Northern Univ. H.S.C.)

20. Explain what is meant by a *modulus of elasticity*.

Distinguish, by means of diagrams, between Young's modulus, bulk modulus and rigidity modulus.

Derive an expression for the bulk modulus of an ideal gas undergoing an isothermal change.

An elastic wire of length 5 metres and diameter 1 mm. is stretched by 0·1 per cent. of its length. Calculate the work done if Young's modulus for the material of the wire is 2×10^{12} dynes per sq.cm.

(Northern Univ. G.C.E. Advanced level.)

CHAPTER 9

PROPERTIES OF FLUIDS : SURFACE TENSION, DIFFUSION AND VISCOSITY

9.1 Surface Effects in Liquids

Flies walking on the surface of a pond are quite a familiar sight in the summer—so also is that of a glass filled so full that the liquid stands higher than the rim of the glass. In both of these cases it seems that the liquid has a 'skin', strong enough to support the weight of a fly, or elastic enough to stretch over the liquid surface in the glass when it is brimming full.

The reality of this skin has often been subjected to criticism, possibly because some descriptions of it have likened it too closely to a sheet of rubber stretched over the liquid. The surface of a liquid does, however, contain a layer of molecules having more than the normal amount of energy and it is shown in the succeeding paragraphs how these molecules can reproduce some of the effects of a stretched membrane.

9.2 Forces on Molecules in a Liquid

The molecules of a liquid are packed sufficiently close together to have some influence on each other, this is unlike a gas (see page 194), where the molecules are so far from each other that they interact only to a very small extent. The 'influence' that one molecule has on another is an attracting force of very short range indeed—the force is quite large when the molecules are close together, but falls off so rapidly that at a distance equal to a few molecular diameters its effects are quite negligible. Around any molecule can be imagined a sphere whose radius is equal to the range of the molecular force, then any molecule which lies within this sphere will be near enough to exert a force on the molecule under consideration (Fig. 9.1).

Fig. 9.1

In general, there will be several molecules at any instant within the sphere, exerting forces of different magnitudes in various directions on the central molecule. These forces can be combined into a resultant force which will accelerate the molecule in a particular direction, and will bring it within the range of another set of molecules. The resultant force on the molecule will change, and it will make a move in yet another direction, whereupon the process will be repeated. The motion

of a molecule will thus consist of a series of short rushes, and since the attracting molecules are distributed at random in the body of the liquid, these rushes will also be of a random nature. This means that a molecule, after making a large number of rushes will still be very near the place at which it started.

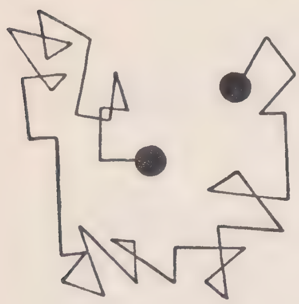


Fig. 9.2

This motion can readily be observed if fine colloidal particles of gamboge (really very large molecules) are suspended in a liquid and viewed under a microscope; the motion of an individual particle is then seen to be somewhat like that depicted in Fig. 9.2. The motion was first observed by Brown, a botanist, about a century ago and is known as *Brownian Motion.*

The fact that the particle does not drift steadily away from its starting position means that over an interval of time the forces acting on it cancel each other out, or the average force acting on the molecule is zero. (Notice the distinction between the resultant force at one instant and the average force over an interval of time.)

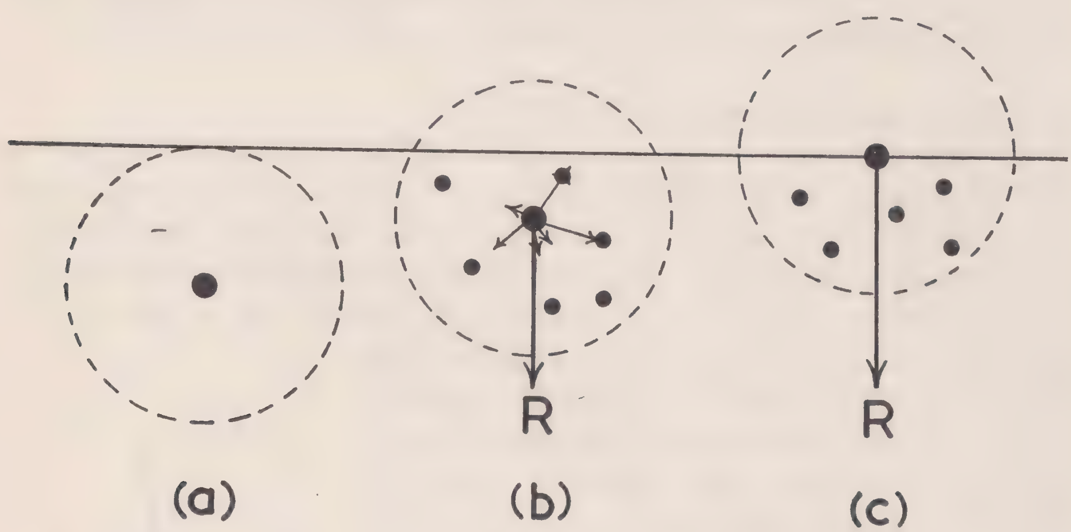


Fig. 9.3

So far the argument has only been applied to a molecule well within the body of the liquid; if the molecule is near the surface as shown in Fig. 9.3 (*b*), then the majority of the molecules influencing the one under discussion must be in the lower half of the sphere associated with it; thus, at any instant, the resultant force stands a bigger chance of being directed downwards rather than upwards, and over an interval of time, the average force will be vertically downwards.

This effect is first apparent when the sphere of influence of the molecule just reaches the surface (Fig. 9.3 (*a*)) and is at its maximum when the molecule is right in the surface (Fig. 9.3 (*c*)); the effect thus extends

only as far down into the liquid as the range of the attractive force of a molecule—some 5 to 10 molecular diameters.

9.3 Free Surface Energy

If a molecule is imagined to make a journey from the body of the liquid to the surface, for the majority of its journey it experiences no average force due to the other molecules. The only work that has to be done on it is the work against its weight and this is just the same as if an isolated molecule were being lifted against gravity; the gain in potential energy is the same in both cases.

As the molecule approaches the surface, however, a downward force is exerted upon it by the other surface molecules; thus, during the last stage of the journey to the surface, an extra amount of work has to be done to move the molecule against this force. A molecule in the surface has, therefore, a small amount of extra energy due to its presence in the surface; its total energy is made up of a certain amount of potential energy, due to its motion against gravity, and some due to its motion against the surface molecular forces, called its *Free Surface Energy.*

In the c.g.s. system of units, surface energy is measured in ergs per square centimetre or erg.cm^{-2}. It represents the amount of work which must be done in order to form one square centimetre of surface and is normally represented by the symbol E, thus

$$\begin{aligned}[E] &= \left[\frac{\text{Energy}}{\text{Area}}\right] \\ &= \left[\frac{ML^2T^{-2}}{L^2}\right] \\ &= [MT^{-2}],\end{aligned}$$

and hence the unit gm.sec^{-2} can also be used for surface energy.

This work assumes that the thermal energy of the molecule is unchanged as it enters the surface.

9.4 Shape of Drops

A mass of liquid possesses potential energy, due to the height of its centre of gravity above some fixed plane, and also surface energy, due to the amount of surface that the mass of liquid possesses. All forms of energy tend to a minimum value as a body takes up a position of stable equilibrium, thus a mass of liquid tends to alter its shape and position so that its total energy is a minimum. In general, it tries to get its centre of gravity as low as possible, and also to become spherical, for the sphere is the figure having minimum surface area for a given volume, in this way it reduces its surface energy to a minimum. (The tendency of a surface to contract can be demonstrated by forming a soap film over the mouth of an inverted funnel. The film slowly rises up inside

the funnel since by so doing it can reduce its area and hence its energy.)

It may not always be possible for both of these effects to occur together; for example, if a layer of liquid is placed on a flat surface, the change to a spherical shape means a lifting of the centre of gravity; this produces a gain in potential energy which may outweigh the reduction in surface energy, and in any practical case the liquid seeks a compromise between the two effects depending on individual circumstances. The following examples serve as an illustration.

(*a*) A raindrop falls towards the Earth to reduce its potential energy and tends to become spherical in order to reduce its surface energy to a minimum.

(*b*) A drop of mercury placed on a horizontal plate becomes spherical if its volume is very small, but if the volume of mercury is increased to form a spherical drop of radius larger than 2 mm, it prefers to spread out into a flattened drop instead (Fig. 9.4.)

Fig. 9.4

Obviously, at a radius of 2 mm, the drop has reached a point where, if it grew any further and remained spherical, the rate of growth of potential energy would outstrip that due to surface energy. The drop therefore chooses to flatten and restrict the rate of growth of potential energy.

(*c*) A volume of liquid placed in a vertical-sided vessel, such as a beaker, reduces its free surface area to a minimum by assuming a flat surface and its potential energy to a minimum by congregating at the bottom of the vessel.

9.5 Surface Tension

It is the tendency of a liquid to reduce its surface area to a minimum, other things being equal, that has led to the idea of a liquid having a *Surface Tension*. The behaviour of the surface is rather similar to that of a balloon made of elastic rubber, which tends to reduce its size so as to release the tension in the rubber. Surface tension is a force generated by the surface of the liquid, and *pulling inwards on the boundary of the surface*. It is important to notice, however, that while the molecules in the surface of a liquid exert sideways forces on each other, the resultant force on a molecule in the surface is downwards into the liquid. There is no resultant force along the surface which can be called the surface tension, except at the boundary of the surface. This is normally demonstrated with a soap film stretched across a wire frame. A piece of cotton tied loosely across the frame will take up any position

on the film and will move around freely on it as long as the film is complete (Fig. 9.5 (*a*)), showing that the cotton is subject to no resultant

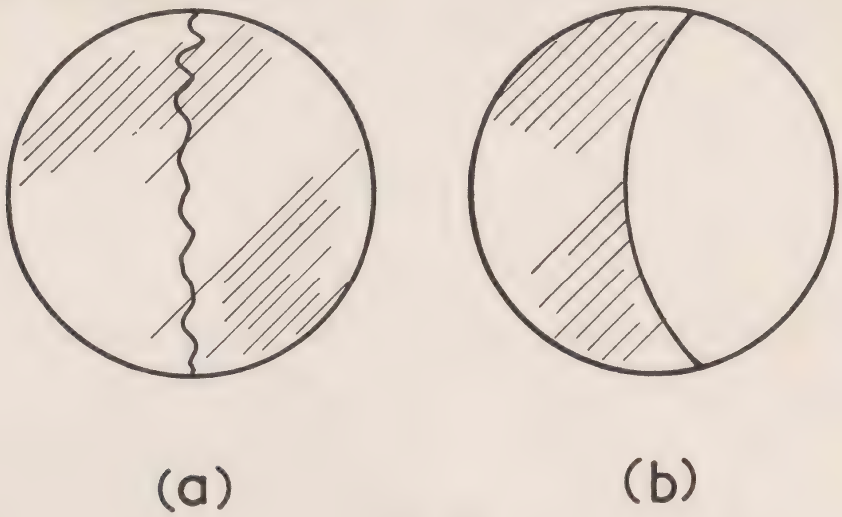


Fig. 9.5

force along the film. If, however, the film on one side of the cotton is broken, it is immediately drawn towards the other side by the tension which now exists along the boundary of the film.

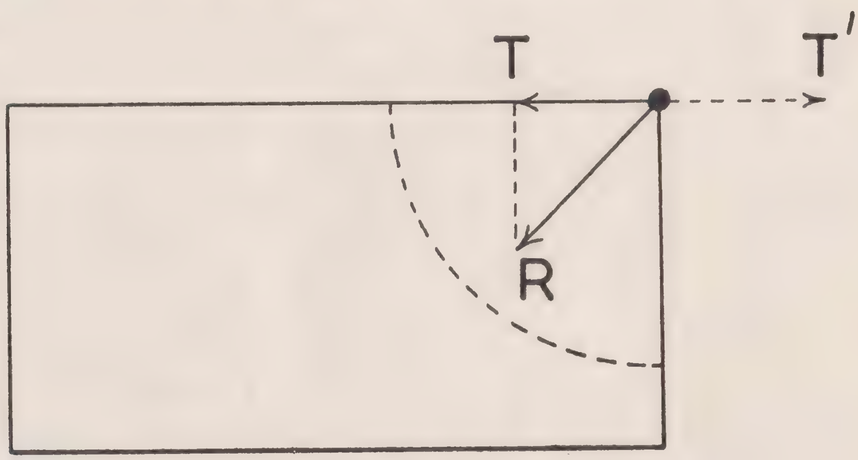


Fig. 9.6

If it were possible to produce a lump of liquid as shown in Fig. 9.6, without the use of a container, the molecule in the top right-hand corner would be attracted only by those molecules lying within the quadrant shown and would experience a resultant force R directed into the body of the liquid. This force has a component T along the surface, and to prevent any horizontal motion of the molecule, an equal and opposite force T' would have to be applied; thus external forces have to be applied to hold the boundary of a liquid surface in position, and it is these forces which are normally equated to the surface tension of a liquid.

The surface tension of a liquid is defined as the force needed to retain all the molecules in one centimetre of boundary in position; surface tension is thus measured in units of force per unit length of boundary. If the symbol S is used for surface tension, then:

$$[S] = \left[\frac{\text{Force}}{\text{Length}}\right]$$

$$= \left[\frac{MLT^{-2}}{L}\right]$$

$$= [MT^{-2}].$$

The c.g.s. unit of surface tension is the gm.sec^{-2}, more commonly known as the dyne.cm^{-1}.

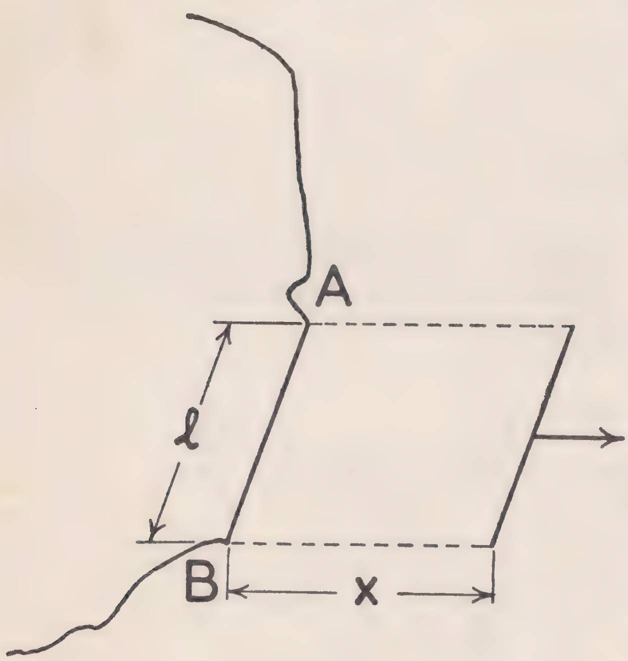


Fig. 9.7

It will be noticed that surface tension and surface energy have the same dimensions and are measured in the same unit (see page 231). The resemblance is even closer than this; for if the surface of a liquid is increased by drawing outwards a straight portion of the boundary as shown in Fig. 9.7, then the inward force along the portion AB of the boundary due to the surface tension S is Sl, and the work done in drawing it out a distance x is Slx. But a new surface of area lx has been formed; thus if the surface energy is E, the gain in energy is Elx. Provided that no other energy change takes place (a change in the thermal energy of the liquid, for example), this gain in energy must come from the work done on the surface, or:

$$Slx = Elx.$$

This equation must be satisfied both dimensionally and numerically; the former has already been seen to be true and the latter can be achieved only if surface tension and surface energy for a given liquid are numerically the same. For water, the value of surface tension is 73 dyne.cm^{-1}, while that for surface energy is 73 erg.cm^{-2}. One important difference between the surface tension of a liquid and the tension in a rubber membrane must be noted. As the rubber membrane is stretched, the tension in it increases, but the surface tension of a liquid remains unchanged however much the surface may be expanded.

9.6 Shape of Meniscus

Normally a liquid must be contained in a vessel, consequently some of the liquid molecules come under the attraction of the molecules forming the wall of the vessel. This force modifies considerably the resultant force acting on the liquid molecules and accounts for the meniscus—i.e. the typical curved surface that a liquid assumes near the walls of a vessel.

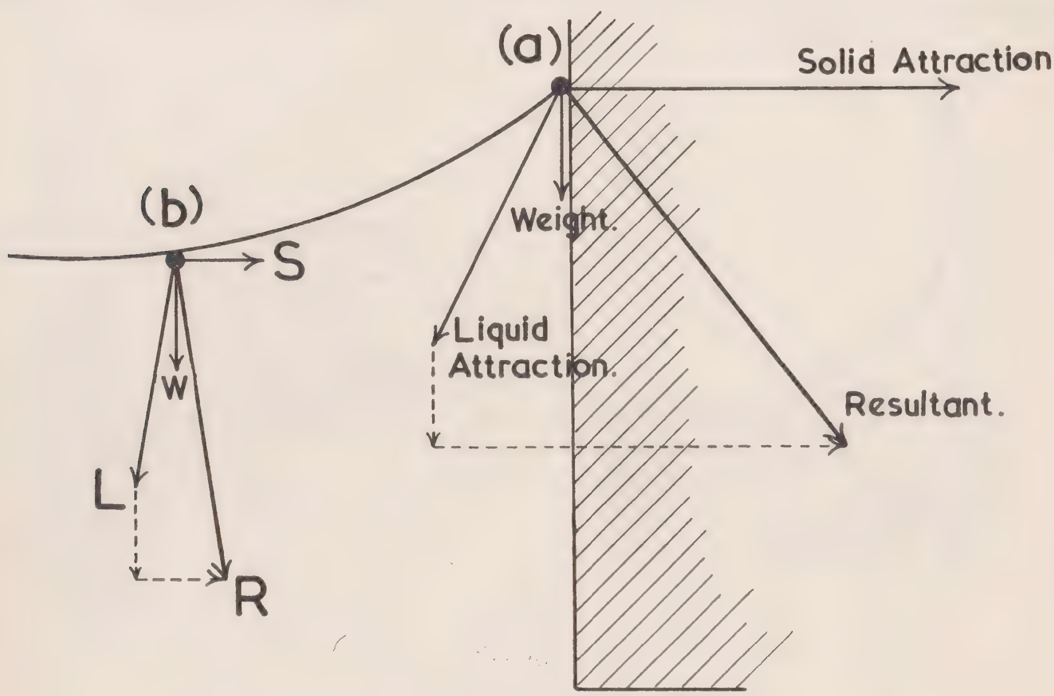


Fig. 9.8

A liquid molecule situated near the wall of a vessel, as at (*a*) in Fig. 9.8, will experience a force directed into the body of the liquid due to the attraction of all the other liquid molecules, also a force directed into the wall due to the attraction of the solid molecules; in addition there is the weight of the molecule acting downwards. Of these three forces, the solid attraction is usually (but not invariably) the greater, consequently the resultant force on the molecule is generally directed into the wall.

Now if the surface of the liquid remained horizontal right up to the wall of the vessel, the resultant force on molecules near the wall would have a component parallel to the surface and pointing towards the wall. This component moves the molecules along and causes them to pile no against the wall, the process continues until the resultant force has up further component parallel to the surface, i.e. until the 'pile-up' near the wall is sufficient to make the surface perpendicular to the resultant force on the molecules.

At points farther removed from the surface (Fig. 9.8 (*b*)) the liquid-

attraction vector becomes more nearly vertical and the solid-attraction vector becomes much smaller, consequently the resultant tends to become more nearly vertical and the surface becomes horizontal.

9.7 Angle of Contact

The angle between the tangent to the surface of a liquid at the point of contact with a wall and the wall itself is called the *Angle of Contact*, illustrated in Fig. 9.9.

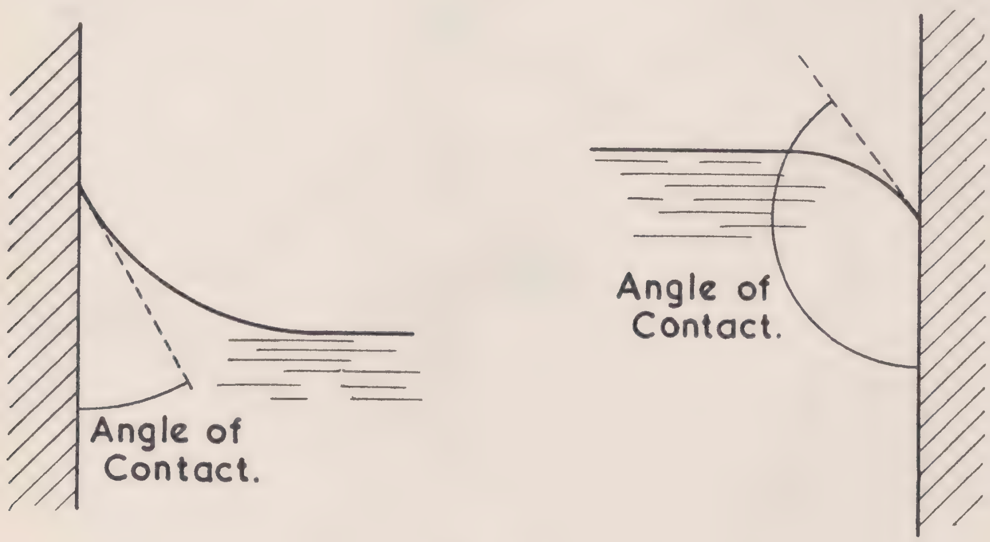


Fig. 9.9

It is evident that the angle of contact depends to a large extent on the magnitude of the attraction exerted by the solid molecules, and this of course depends on the material of the walls; thus each liquid does not have a unique value for its angle of contact, and values can be quoted only for a given liquid and solid combination. For example, the angle of contact of water at a glass wall is zero, while at a chromium wall the angle of contact is about 160°. In practice, angles of contact ranging from 160° down to 20° are found and also some which are equal to zero. If the angle of contact is zero for a given combination, then the liquid is said to 'wet' the solid; if the level of the liquid in a vessel is lowered, a film of liquid will be left on the walls if they are wetted by the liquid, but if the angle of contact has any finite value, then the walls are left dry instead.

To summarise, then, a liquid has a specific value of surface tension, measured in dyne.cm^{-1}, which is numerically equal to its surface energy measured in erg.cm^{-2}; but the angle of contact between the liquid and a wall depends also on the material of the wall. In general, contact angles varying from 160° down to 20° are found, or the liquid wets the wall, in which case the angle of contact is zero.

9.8 Methods of Measuring Surface Tension

(a) Torsion Balance

The obvious method of finding the surface tension of a liquid is merely to measure the force needed to retain a known length of surface boundary in position.

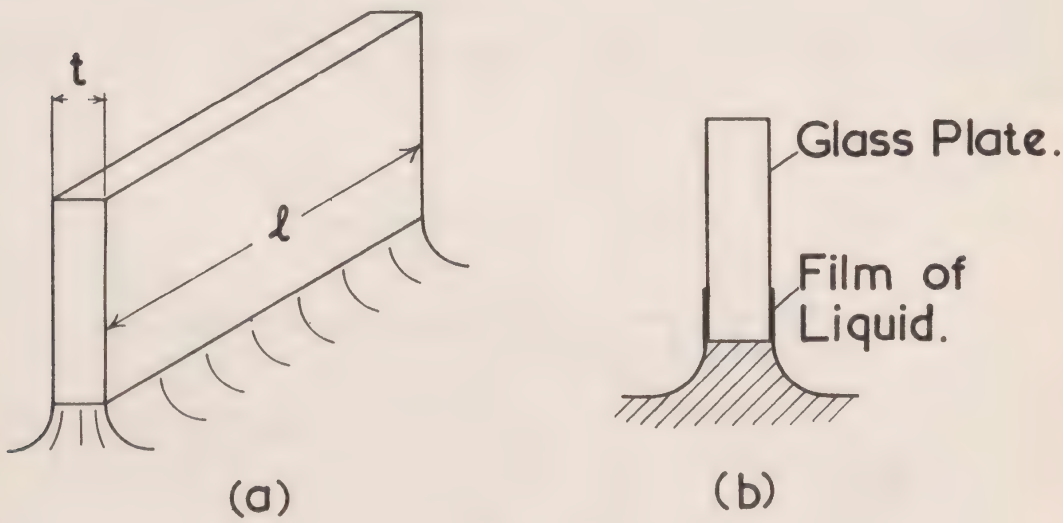


Fig. 9.10

Forces are best measured when they act vertically, for then they can be measured by weighing with a balance; this means that the surface of the liquid must be attached to a boundary capable of motion in the vertical direction and is usually done by dipping into the liquid a glass microscope slide held on edge. As the slide is drawn out of the liquid (Fig. 9.10 (*a*)), a meniscus is formed attached to the bottom edge as shown, and a liquid surface boundary runs all around the lower rim of the slide. If the liquid wets the material of the plate, then the meniscus will be vertical where it joins the plate (Fig. 9.10 (*b*)) and the surface-tension forces exerted on the slide will be vertically downwards. If the slide is of length l and thickness t the length of the surface boundary attached to it is $2(l + t)$, and if the surface tension of the liquid is S, the downward force due to surface tension is $2(l + t)S$.

Thus to hold the plate in equilibrium in this position, an upward force equal to $[2(l + t)S + mg]$ must be applied to it, where mg is the weight of the plate itself. This can be done by hanging the plate from one arm of a balance and adding weights by very small increments to the other pan. When these weights exceed $[2(l + t)S + mg]$, the surface tension will no longer be able to hold the plate down and the meniscus will break; it is thus necessary to find the maximum load which can be placed in the other pan without breaking the surface film. If this load is provided by a mass M, then:

$$2(l + t)S + mg = Mg$$

$$\text{or } S = \frac{(M - m)g}{2(l + t)} \quad . \quad . \quad . \quad (1)$$

The ordinary balance is not an ideal instrument for making such a measurement, owing to the small jerk which must occur every time a weight is added to the pan—this may cause a premature breaking of the meniscus. A balance is needed in which the load may be increased continuously without jerks; such a balance is the torsion balance (Fig. 9.11).

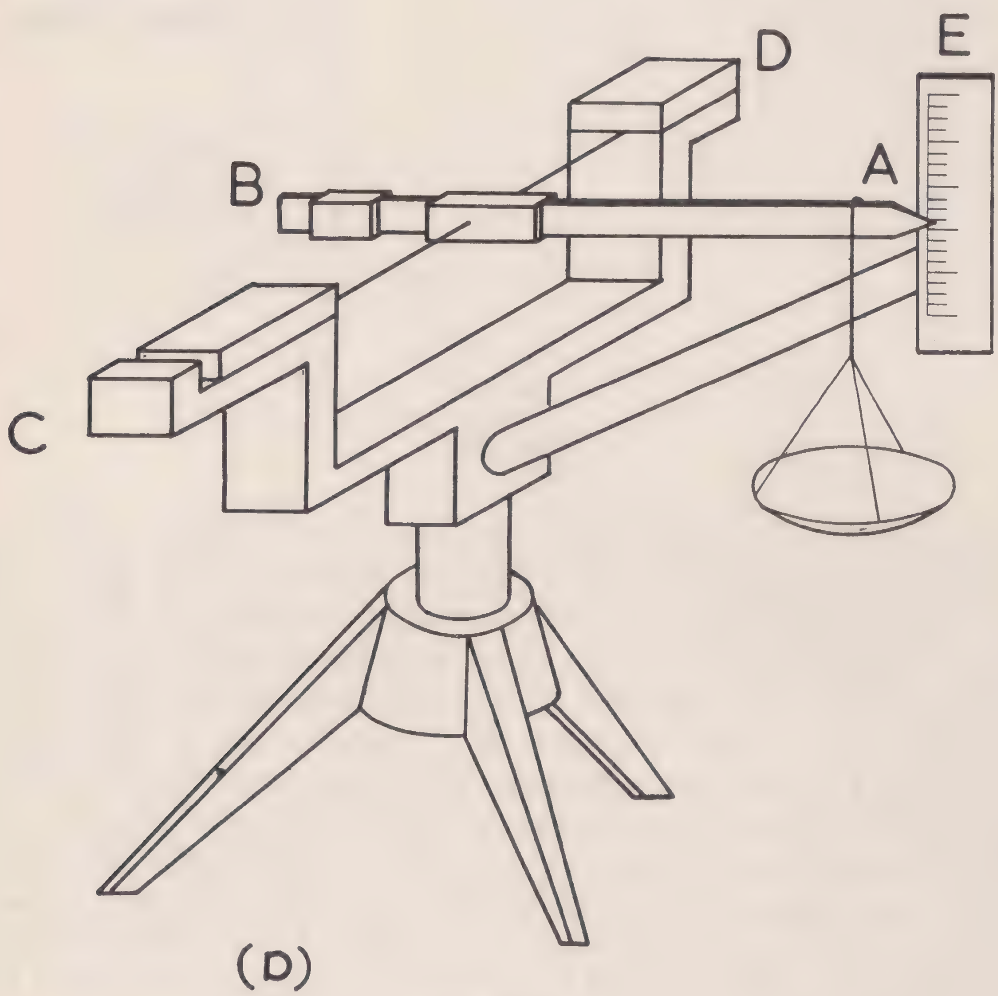


Fig. 9.11

This consists of a beam AB rigidly fixed to a wire which is stretched between two supports C, D. If the end A of the beam is depressed, the wire will twist and will generate a couple opposing the motion of the beam. The couple produced by a wire in torsion is proportional to the twist and so can be made to grow very steadily if the twist in the wire is gradually increased. For details of this method, the student is referred to *Experimental Physics*, by Daish and Fender, published by English Universities Press, Ltd.

(b) Capillary Rise

The forces due to surface tension account for the fact that, if a tube is stood in a vessel of liquid, the liquid rises up inside the tube above the general level of the liquid outside the tube, Fig. 9.12.

The case can be examined as follows. All round the meniscus inside the tube, the molecules of the wall will exert forces on the molecules of the liquid and these forces will be equal and opposite to the surface tension forces if the meniscus is in equilibrium. If the liquid wets the walls of the tube, these forces will act in the vertical direction; some of them are shown as T in the diagram. If the surface tension of the liquid is S and the radius of the tube is r, then the length of surface boundary is $2\pi r$ and the total upward surface tension force is $2\pi rS$. This force must support the weight of the column of liquid inside the tube, thus if its height is h, the weight of the column is $\pi r^2 h\varrho g$ (where ϱ is the density of the liquid), and:

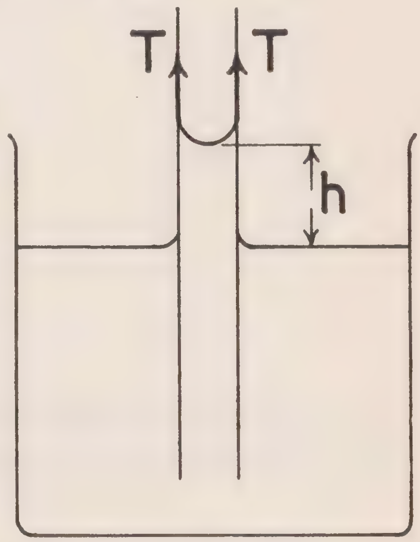


Fig. 9.12

$$2\pi rS = \pi r^2 h\varrho g \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{or } S = \frac{rh\varrho g}{2} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This method again suffers from the neglect of the weight of the liquid in the meniscus, i.e. the part shaded in Fig. 9.13, but the correction for

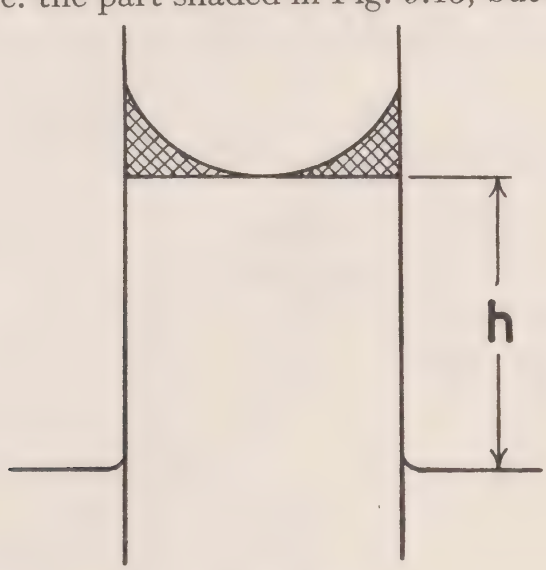


Fig. 9.13

the meniscus is accurately known for any size of cylindrical tube, and when this and other corrections are applied the method becomes one of high precision.

If the angle of contact is not zero, the component of the surface tension forces in the vertical direction becomes $S \cos \theta$ (Fig. 9.14).

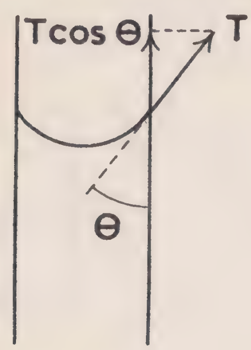


Fig. 9.14

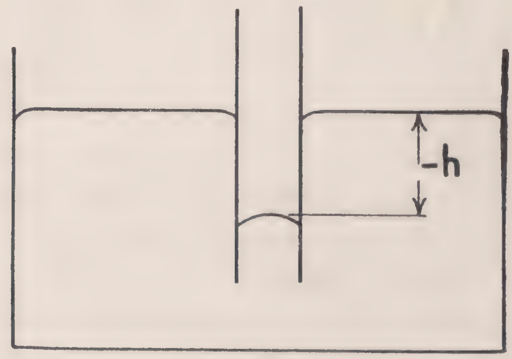


Fig. 9.15

If this value is substituted for S in Equation (3), the expression for the surface tension becomes:

$$S = \frac{rh\rho g}{2 \cos \theta} \quad . \quad . \quad . \quad . \quad (4)$$

This equation indicates that, if the angle of contact is greater than 90°, the liquid in the tube will be depressed as in Fig. 9.15. In this case, $\cos \theta$ has a negative value, but the surface tension must have a positive value, hence h must take a negative value if Equation (4) is to be satisfied.

(c) Shape of Threads

The shape taken up by a thread when drawn to one side by a soap film was illustrated in Fig. 9.5 (*b*). This case can now be investigated more fully.

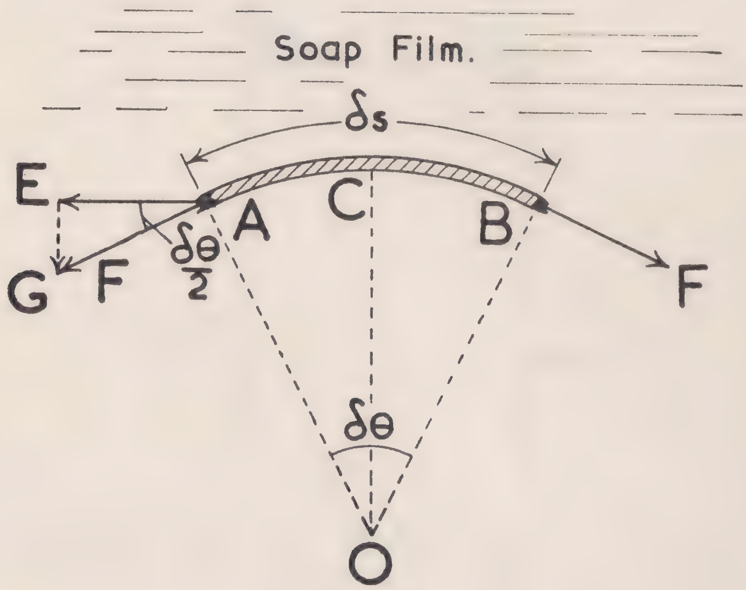


Fig. 9.16

Consider an element of the thread (Fig. 9.16) of length δs, so small that it can be considered as an arc of a circle, centre O and of radius r. Then δs will subtend an angle $\delta\theta$ at O given by:

$$\delta s = r\,\delta\theta \quad . \quad . \quad . \quad . \quad (5)$$

A uniform tension F in the cord will act at the ends of the element as shown; if C is the midpoint of the element, draw AE perpendicular to OC and resolve the left-hand force into vectors AE and EG along and perpendicular to AE respectively. Then:

$$AE = F\cos\frac{\delta\theta}{2}$$

$$\text{and } EG = F\sin\frac{\delta\theta}{2} \quad . \quad . \quad . \quad . \quad (6)$$

The right-hand force F will resolve into a component $F\cos\frac{\delta\theta}{2}$, opposing the force along AE, and thus producing equilibrium in this direction, and a force $F\sin\frac{\delta\theta}{2}$ acting in the same direction as that along EG.

Now EG is parallel to CO, therefore the total force in this direction is given by:

$$\text{Force} = 2F\sin\frac{\delta\theta}{2}$$

$$= 2F\left(\frac{\delta\theta}{2}\right)$$

if $\delta\theta$ is sufficiently small for $\frac{\delta\theta}{2}$ to be approximately equal to $\sin\frac{\delta\theta}{2}$.

$$\text{Thus force} = F\,\delta\theta$$

$$= \frac{F}{r}\,\delta s \text{ (substituting from Equation (5)).}$$

The force per unit length acting on the element in a direction parallel to CO is therefore equal to F/r.

Now if the thread is in equilibrium, the resultant force on it must be zero; thus the soap film must exert on the thread a force equal and opposite to that due to the tension in the thread.

If S is the surface tension of the soap solution, the force exerted by the film on unit length of boundary is $2S$ (since a soap film has a surface on each side), hence for equilibrium:

$$F/r = 2S$$

$$\text{or } S = F/2r \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Notice that the surface tension of the film has a constant value at all points and the tension in the thread must be the same all the way along, hence if Equation (7) is to be satisfied, r must be a constant for every element of the thread; it is therefore drawn into an arc of a circle of radius r.

A method of measuring surface tension depending on the foregoing theory can be developed as follows.

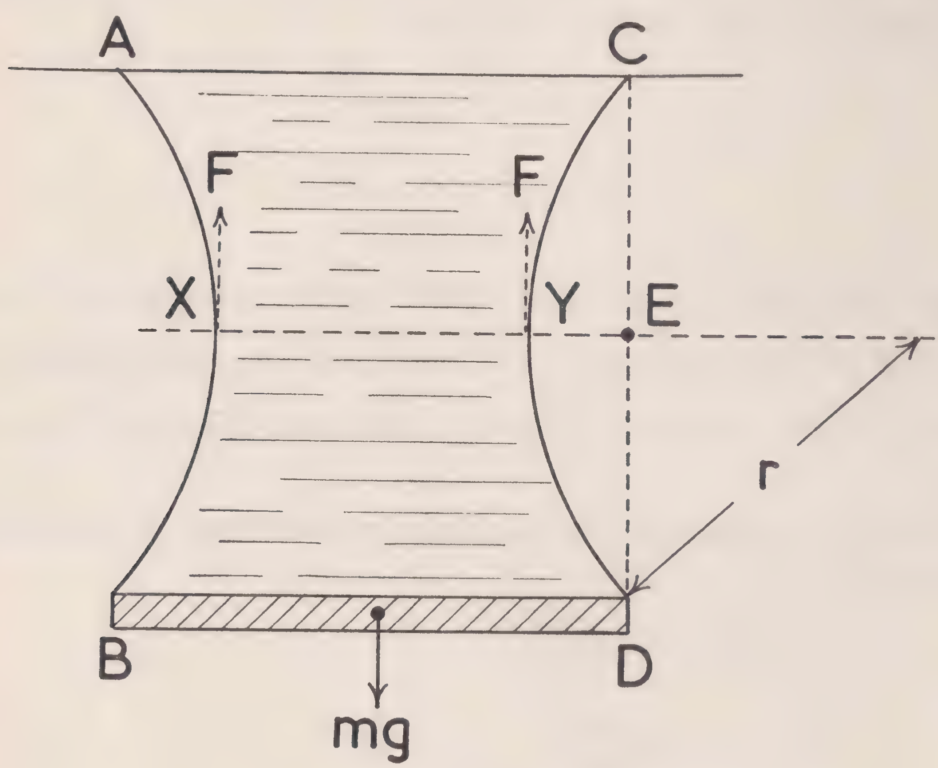


Fig. 9.17

A bar of mass m is suspended horizontally by two equal vertical threads AB, CD (Fig. 9.17), a soap film is then formed over the threads drawing them into arcs of circles as shown in the figure. If the threads remain vertical at the points X and Y, then, ignoring the weight of the film itself, the equilibrium of the part of the apparatus below XY is given by:

$$2F = mg,$$

where F is the tension in either thread.

Substituting from Equation (7) gives :

$$4rS = mg,$$

$$\text{thus } S = mg/4r,$$

$$\text{but } r = \tfrac{1}{2}\left\{EY + \frac{ED^2}{EY}\right\},$$

$$\text{thus } S = mg/2(EY + ED^2/EY) \quad . \quad . \quad . \quad (8)$$

Example 1. *A plane soap film is formed over a wire frame and a fine rubber band, which in its unstretched condition forms a circle of diameter 5 cm, is dropped on to the film. The film inside the rubber band is broken, whereupon the band is stretched into a circle of diameter 5·4 cm. It is also found that the band, when cut and used as a single strand, is stretched to double its length when suspending a 2-gm weight. Find the surface tension of the soap film.*

If the band is perfectly elastic, then it will obey Hooke's Law, i.e.

$$F = kX$$

where F is the tension in the band and X its extension. The value of k can be found by substituting numerical values from the second part of the experiment,

$$\text{thus } 2 \times 981 = k \times 5\pi$$
$$\text{or } k = \frac{2 \times 981}{5\pi}.$$

This enables the tension F_1 in the band during the first part of the experiment to be calculated, since:

$$F_1 = \frac{2 \times 981}{5\pi} \times 0{\cdot}4\pi \text{ dynes}$$
$$= 0{\cdot}16 \times 981 \text{ dynes}$$

and using Equation (7), this gives the surface tension of the film:

$$S = \frac{0{\cdot}16 \times 981}{5{\cdot}4} \text{ dyne.cm}^{-1}$$
$$= 29 \text{ dyne.cm}^{-1}.$$

9.9 Excess Pressure inside a Bubble

(a) Spherical Surfaces

A small excess pressure has to be maintained inside a bubble to counteract the tendency of the liquid surface to contract. It is possible to calculate the pressure inside any shape of bubble, but the treatment which follows is applicable only to a spherical bubble. It is also necessary to distinguish between two sorts of bubbles—those blown of a gas under a liquid, which have only one liquid surface, and soap bubbles having a liquid surface both on the inside and outside of the film.

The bubble can be thought of as split into two hemispheres by a diametral plane (Fig. 9.18); the pressure inside the bubble tries to blow the two halves apart, while the surface tension forces around the join at the diameter hold the two halves together; if the two halves of the bubble are at rest, these two effects must be in equilibrium, i.e. must exert equal and opposite forces.

Let the excess pressure inside the bubble be p; this exerts an outward force on every element of the surface, but only the component of each force perpendicular to the diametral plane tends to push the halves apart.

Consider an element $ABCD$ of area δa. The outward force on this is $p\delta a$ (Fig. 9.19) and the component perpendicular to the diametral plane is $p\delta a \sin\theta$, thus the total force pushing the two halves apart is $\Sigma p\delta a \sin\theta$. Now p is a constant, hence this force may be written as

$p\Sigma\delta a \sin\theta$; also $\delta a \sin\theta$ is the projected or resolved component of the area $ABCD$ parallel to the diametral plane, i.e. the area $A'B'C'D'$.

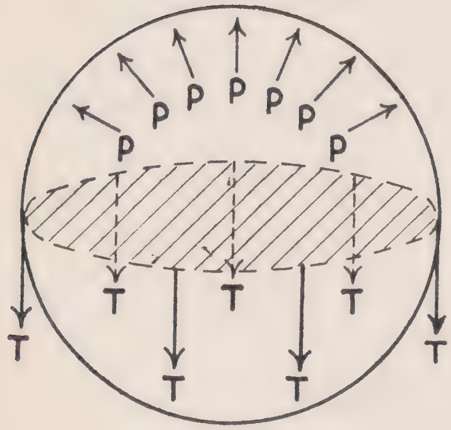


Fig. 9.18

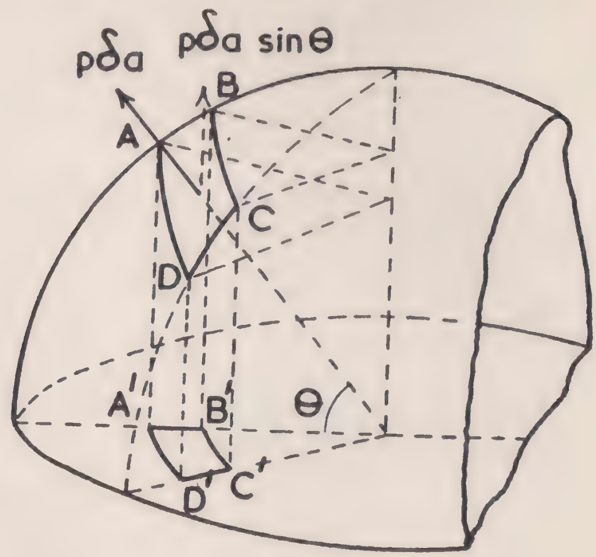


Fig. 9.19

(Notice that area is a vector quantity, see page 11, and so can be resolved into components as any other vector.)

The total force is therefore given by:

$$\text{Force} = p\ \Sigma\ \text{Area } A'B'C'D',$$

where the summation has to cover all areas such as $A'B'C'D'$ corresponding with every element of area of the surface of the hemisphere.

Σ Area $A'B'C'D'$ is thus equal to the area of the diametral plane, i.e. πr^2, and the force separating the hemispheres is given by:

$$\text{Force} = p\pi r^2.$$

The two hemispheres are held together by the surface tension forces acting around the rims; the boundary of the hemisphere is of length $2\pi r$, hence, if the surface tension of the liquid is S and the bubble is a 'single-surface' one, i.e. a bubble of gas blown under liquid, the surface tension force around the boundary is $2\pi rS$; thus for each hemisphere of the bubble to be in equilibrium:

$$2\pi rS = p\pi r^2$$

$$\text{or } p = \frac{2S}{r} \qquad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

If the bubble is a soap bubble, having an inside and an outside surface, both of radius r, the surface tension force due to each surface is $2\pi rS$ and the total surface tension force is $4\pi rS$. For equilibrium, therefore:

$$4\pi rS = p\pi r^2$$

$$\text{or } p = \frac{4S}{r} \qquad . \quad . \quad . \quad . \quad . \qquad (10)$$

This result is in some ways rather surprising, as it indicates that the pressure in a bubble *decreases* as the size of the bubble increases, but it can be demonstrated to be true if two bubbles of different size are blown at opposite ends of a capillary tube (Fig. 9.20); if the tap is opened it is found that the small bubble decreases in size and increases the larger one (Fig. 9.21).

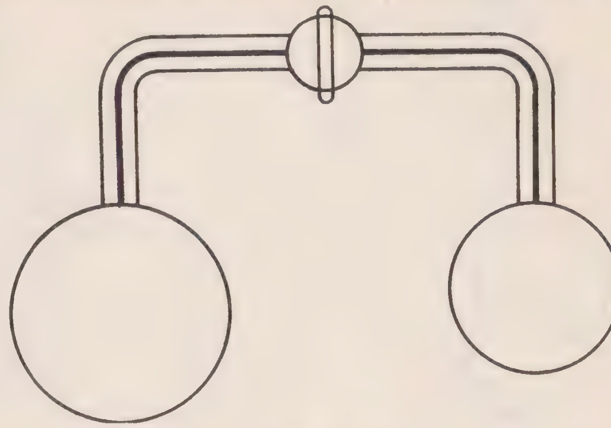


Fig. 9.20

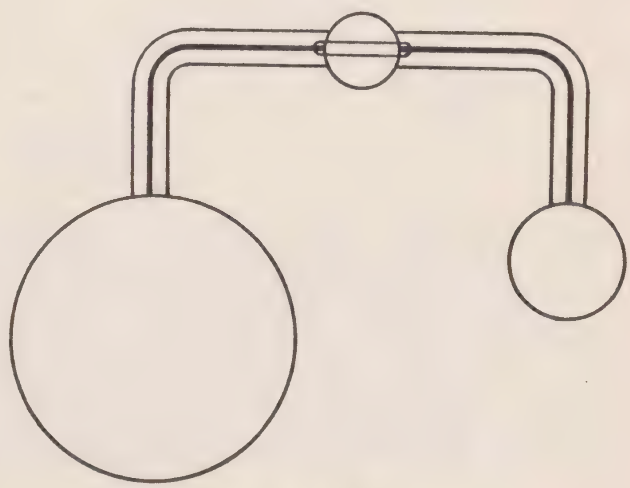


Fig. 9.21

(b) Non-spherical Surfaces

Many curved surfaces are obviously not parts of spheres—a barrel provides a good example; in its construction the staves are first bent along their length to a radius of curvative r_1 (Fig. 9.22 (*a*)), and they are then fitted together so that they can be embraced by a circular hoop of radius r_2. At a point such as X on the surface of the barrel (Fig. 9.22 (*b*)) the surface therefore has two radii of curvative r_1 along the length of the staves and r_2 around the circumference of the barrel.

The curvature of all curved surfaces can be expressed in terms of two radii at any point; these are called the principal radii of curvature of the surface. The magnitude of the radii may vary from point to point on the surface.

It is possible to calculate the excess pressure needed to maintain a non-spherical bubble in terms of its principal radii; the work is rather beyond the standard of this book, but if it is done, Equations (9) and (10) become:

Excess pressure in a 'single-surface' bubble formed under liquid

$$= S\left(\frac{1}{r_1}+\frac{1}{r_2}\right) \quad . \quad . \quad . \quad . \quad (11)$$

Excess pressure in a soap bubble

$$= 2S\left(\frac{1}{r_1}+\frac{1}{r_2}\right) \quad . \quad . \quad . \quad . \quad (12)$$

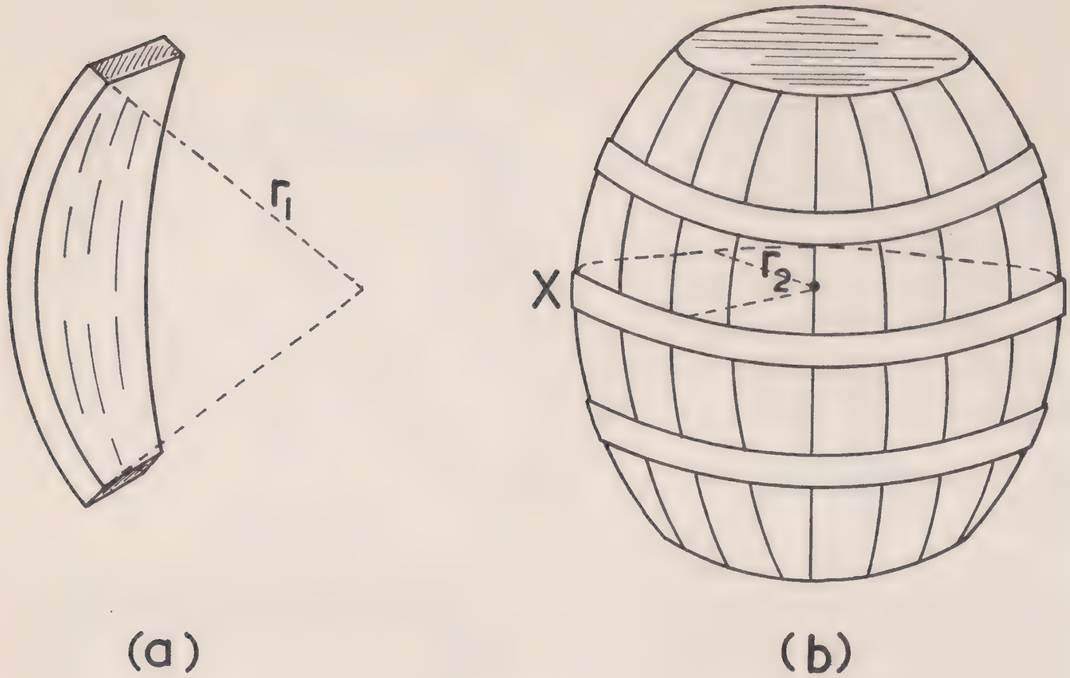


Fig. 9.22

The principal radii r_1 and r_2 are to be interpreted as positive if the centre of curvature lies on the same side as the excess pressure, i.e. 'inside' the bubble, and negative if it lies outside.

It will be noted that if the surface is spherical, the two principal radii are both equal to the radius of the sphere; with this substitution, Equations (11) and (12) reduce to (9) and (10) respectively.

9.10 Methods of Measuring Surface Tension based on Excess Pressure inside a Bubble

(a) Jaeger's Method

If the pressure in a bubble of given radius formed under a liquid is measured, then the surface tension of the liquid can be calculated by using Equation (9). Jaeger's method is basically this, although the radius of the bubble and the corresponding pressure are measured in a rather indirect fashion.

The bubble is blown from a tube dipping into the liquid; consequently when the bubble first appears it is of very large radius (Fig. 9.23 (*a*)), but this radius gradually *shrinks* as the bubble develops, Fig. 9.23 (*b*)). The radius reaches its minimum value when the bubble is hemispherical (Fig. 9.23 (*c*)), and then grows again as the size of the bubble increases (Fig. 9.23 (*d*) and (*e*)).

Remembering that the excess pressure inside the bubble is given by $p = 2S/r$, it will be seen that the pressure reaches a maximum value when the bubble has minimum radius, i.e. is hemispherical.

It is impossible to experiment on bubbles as shown in Fig. 9.23 (*d*) and (*e*). If the pressure of the source of compressed gas used to blow the bubbles is gradually increased, the bubble will pass through the

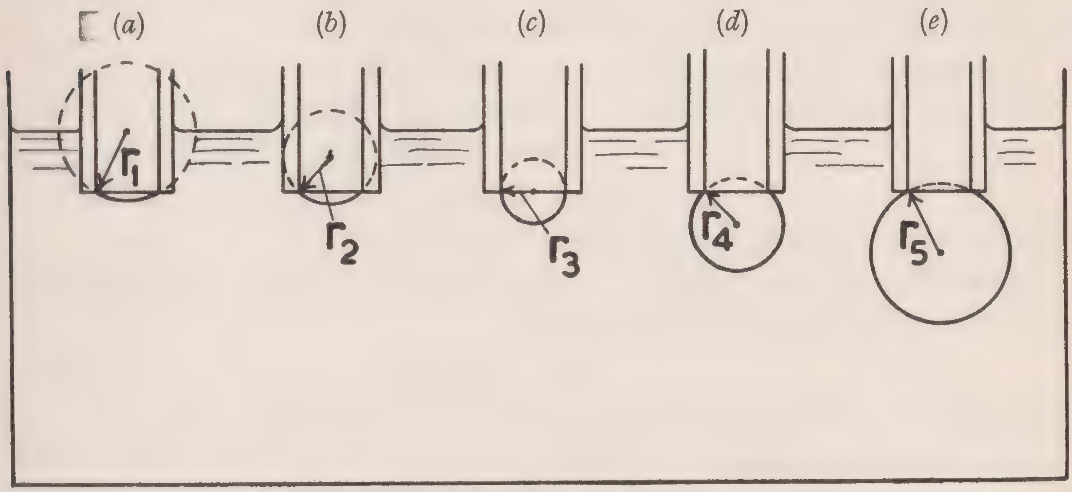


Fig. 9.23

stages (*a*) and (*b*) to (*c*). Any further increment in pressure will enlarge the bubble beyond the hemispherical shape, increase its radius, and so *reduce* the pressure needed to maintain the bubble. The pressure of the source, however, does not reduce; therefore the bubble expands still further, aggravating the condition, and finally bursts. All this, of course, happens very quickly immediately after the bubble passes through the hemispherical stage; the bubble then detaches itself from the tube and pops up to the surface.

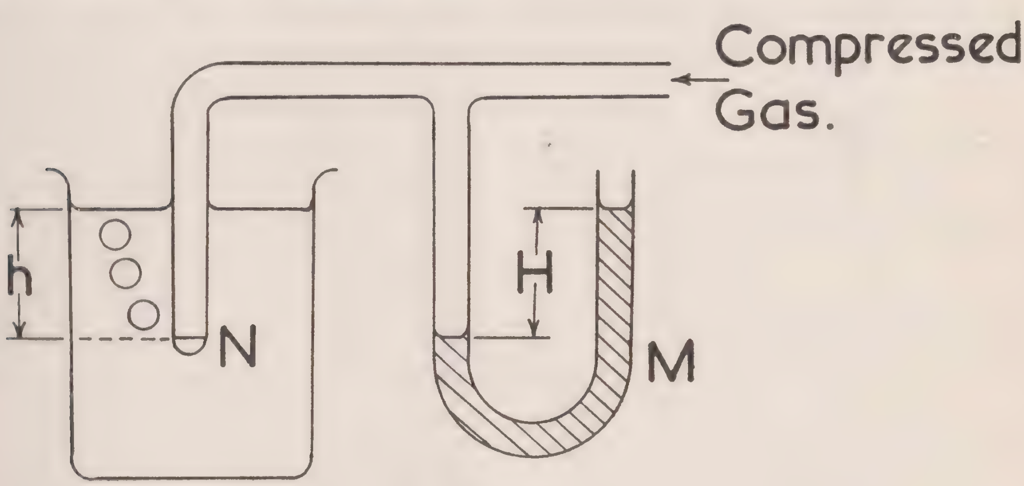


Fig. 9.24

The bubbles are blown with an apparatus as shown in Fig. 9.24. The nozzle N, used to blow the bubbles, is a tube with its end ground flat; the pressure inside the system is measured with the manometer M. As the bubble approaches the hemispherical form, the pressure in it rises to a maximum and then drops sharply as the bubble breaks away.

This shows on the manometer as a gradual climb to a maximum reading (when the bubble is hemispherical), followed by a sharp drop as the bubble breaks. The *maximum* reading of the manometer therefore records the pressure in the bubble when it is hemispherical and of the same radius as the inside radius of the nozzle.

The theory of the method can be developed as follows.

Pressure outside bubble = atmospheric pressure + hydrostatic pressure due to submersion in liquid.

$$= p_0 + g\varrho_1 h$$

where ϱ_1 is the density of the liquid under test.

Pressure inside the bubble = atmospheric pressure + hydrostatic pressure shown on the manometer

$$= p_0 + g\varrho_2 H,$$

where ϱ_2 is the density of the liquid in the manometer.

Thus excess pressure in bubble

$$= (p_0 + g\varrho_2 H) - (p_0 + g\varrho_1 h)$$
$$= g(\varrho_2 H - \varrho_1 h).$$

But if the bubble is of radius r, the excess pressure is equal to $2S/r$, where S is the surface tension of the liquid.

$$\text{Hence } \frac{2S}{r} = g(\varrho_2 H - \varrho_1 h)$$

$$\text{or } S = \frac{rg}{2}(\varrho_2 H - \varrho_1 h) \quad . \quad . \quad . \quad . \quad . \quad (13)$$

(b) Force required to separate Wetted Plates

It is common experience that if two wet glass plates are placed together it is very difficult indeed to get them apart again. This effect can be attributed to the surface tension of the liquid as described below. For simplicity, consider two circular plates of radius R placed together with a film of liquid between them. If the liquid wets the solid, then as the plates are pulled apart, the surface of the liquid will be shaped as shown in Fig. 9.25. If the thickness of the liquid layer is $2r$, the principal radii of curvature of the film will be very nearly R and $-r$ (considered from inside the liquid film), consequently the excess pressure inside the liquid is given by:

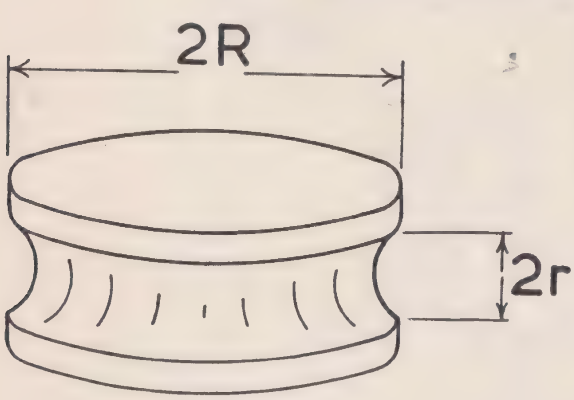


Fig. 9.25

$$p = S\left(\frac{1}{R} - \frac{1}{r}\right).$$

In general, r is much smaller than R, hence this expression gives a negative excess pressure, or a lack of pressure, inside the film. The plates are thus 'sucked' together by the lower pressure in the liquid film. The force pressing the plates together is equal to pA, where A is the area of one plate, or:

$$\text{Force holding plates together} = -S\left(\frac{1}{R} - \frac{1}{r}\right) . \pi R^2$$

$$\simeq \frac{\pi R^2 S}{r}$$

if $1/R$ can be ignored in comparison with $1/r$.

The magnitude of this force is usually very large because of the thinness of the film which forms between the two plates; for example, if two discs 3 cm in diameter (about the same size as a penny) are placed together with a layer of water 0·01 mm thick between them, then the force needed to separate the plates by a direct pull is given by:

$$F = \frac{\pi \times (\frac{3}{2})^2 \times 73}{0{\cdot}0005} \text{ dynes}$$

$$\simeq 1 \text{ kg-wt.}$$

9.11 Evaporation and Latent Heat

In the earlier sections of this chapter it was shown that a certain amount of work had to be done on a molecule to move it from the body of a liquid to the surface. The position is worsened if the molecule is to be removed from the liquid altogether, i.e. is to be evaporated from the surface. Work must be done on the molecule to get it into the surface and further work to get it beyond the surface, for the downward force which begins to act on the molecule when in the position (*a*), Fig. 9.26, does not cease until the molecule reaches the position (*b*). The energy needed to overcome this force is normally extracted from the thermal energy possessed by the liquid, and thus evaporation of molecules causes cooling of the liquid. Even the formation of a fresh liquid surface causes a cooling of the liquid for the same reason. If the liquid is to remain at a constant temperature whilst evaporation takes place, then heat must be supplied, and the amount needed to maintain the temperature while all the molecules in one gram of the liquid evaporate is called the *latent heat of vaporisation*.

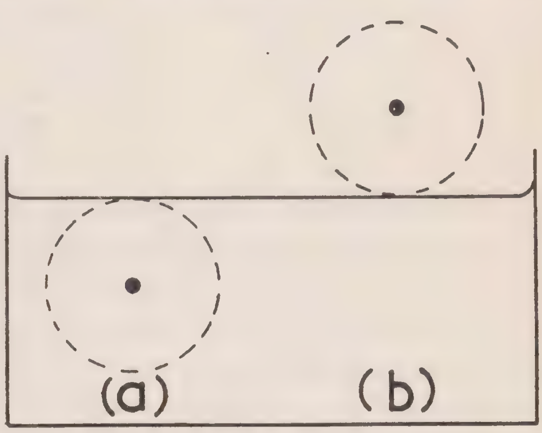


Fig. 9.26

9.12 Vapour Pressure

The rate at which molecules evaporate depends mainly on the temperature of the liquid, for a molecule can be evaporated only when it has enough kinetic energy to overcome the restraining forces as it passes through the liquid surface. In general, the kinetic energy of molecules is distributed about some mean value determined by the temperature, as shown in Fig. 9.27. At a temperature T_1 the average kinetic energy of the molecules is E_1 and so on. There are always a few molecules with energy enough to escape (i.e. greater than E_{min}), and this number increases steadily as the temperature of the liquid goes up.

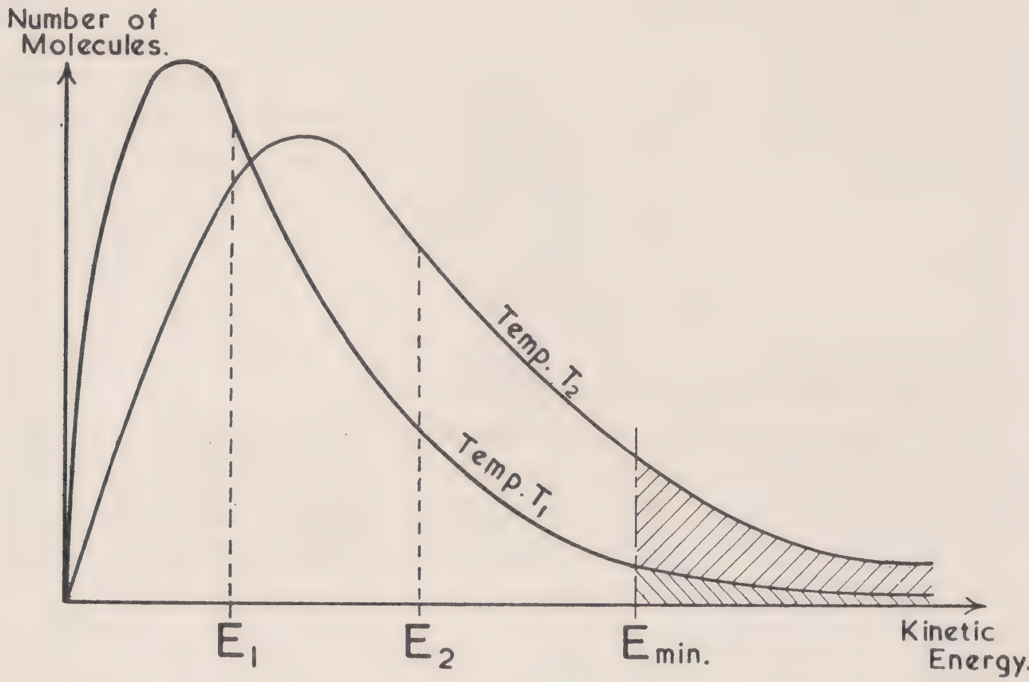


Fig. 9.27

As the evaporation proceeds, the space above the liquid gradually becomes filled with liquid molecules in the gaseous state; this is called a vapour, and the pressure in the vapour space gradually increases as the concentration of molecules goes up. It was seen in Chapter 7, however, that the pressure in a gas is due to the impacts of the molecules, thus as the vapour pressure builds up, so the rate at which the vapour molecules collide with the liquid surface also increases.

Now once a molecule has collided with the liquid surface, it cannot escape again until it picks up sufficient kinetic energy to be evaporated —i.e. for the time being it has condensed back into a liquid; the process is thus a continuous one, some molecules being evaporated from the liquid and at the same time some condensing on the liquid.

If the rate of condensation of molecules is equal to the rate of evaporation, the same number of molecules leave and enter the vapour

space; it is then said to be *saturated* and the pressure in it is described as the *saturation vapour pressure.*

Notice that the saturation vapour pressure occurs when the condensation rate equals the evaporation rate. This is governed by the kinetic energy of the liquid molecules, which in turn depends on the temperature of the liquid; thus the saturation vapour pressure depends mainly on the temperature of the liquid.

The evaporation rate also depends to a small extent on the shape of the liquid surface as will be seen from Fig. 9.28. This diagram illustrates five stages in the passage of a molecule through a plane and convex surface respectively; it has been constructed using the idea that the downward force on a molecule is due to the molecules in the lower half of its 'sphere of influence' (shaded in the diagram), less the effect due to the molecules (if any) in the top half of the sphere of influence (cross-hatched in the diagram).

It will be seen that initially the downward force on a molecule approaching a convex surface is greater than if it approaches a plane surface, but once it gets fairly close to the convex surface, and after it has passed through it, the retarding force is always smaller than when negotiating a flat surface. The result of this variation in force is that the energy needed to escape from a convex surface is lower than that needed for a flat surface. Hence, at a given temperature, the rate of evaporation of molecules from the convex surface exceeds that from the flat surface and consequently can build up a greater saturation vapour pressure. It can be shown that the increase in vapour pressure δp is given by:

$$\delta p = \frac{\sigma}{\varrho - \sigma} \cdot \frac{2S}{r} \qquad . \quad . \quad . \quad . \quad (14)$$

S is the surface tension of the liquid, ϱ its density, σ the density of the vapour and r the radius of curvature of the surface.

The effect of curvature on vapour pressure is not large, in fact at normal temperatures the size of a drop of water has to be reduced to about 0·00004 cm diameter in order to increase the saturation vapour pressure by 1 per cent.

If a vapour is saturated over a liquid, then droplets cannot exist in the vapour. The space will not be saturated with respect to the curved surface of the drops and they will evaporate—this will super-saturate the vapour space with respect to the flat surface of the liquid and the excess vapour will condense on it.

Further, drops cannot grow spontaneously in a vapour. It will be seen from Equation (14) that if a drop is to grow from zero radius, an infinite degree of supersaturation would be needed to start it off. In fact, for any given degree of supersaturation, Equation (14) gives the minimum size of drop which can exist. In a region with a given

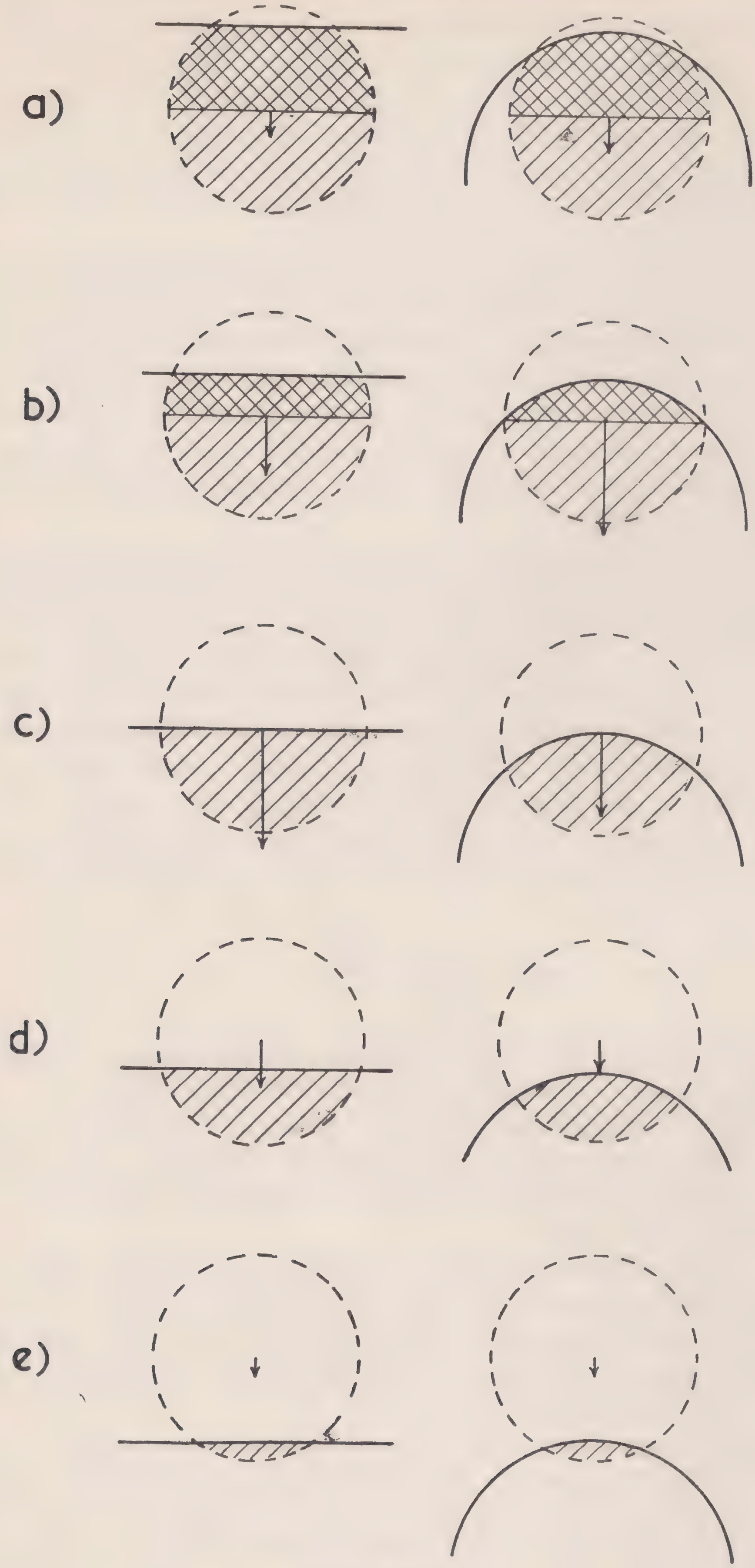


Fig. 9.28

degree of supersaturation, drops will start to form only if nuclei larger than this minimum size are present. Particles of dust or salt floating in the air usually provide these nuclei and may account for the preponderance of fogs and mist found in large industrial areas with smoke-laden atmospheres.

9.13 Experimental Work

One warning must be given concerning experiments designed to measure surface tension. If any foreign molecules are present in the liquid (this must always be the case however pure the liquid), they will generally find their way into the surface of the liquid. This happens because the attraction between liquid molecules and most dissolved molecules is less than the attraction between the liquid molecules themselves. Thus less work is needed to push the solute molecules into the surface, i.e. the surface energy can be minimised if the surface layer is formed of impurities.

This does not increase the experimenter's chances of measuring the surface tension of the liquid—most of his experiments are carried out on the skin of impurities and so give a low value. Experimental methods have been devised to break up this layer, Jaeger's method is an example, but most employ rather advanced techniques. The position has degenerated considerably since the advent of the modern detergents—these cause a drastic reduction in the surface tension of water even when present in the smallest concentration. Laboratory glassware, once washed in such a detergent, should be rejected for experiments on surface tension, however well it may subsequently have been rinsed.

9.14 Diffusion

A crystal of potassium permanganate placed at the bottom of a beaker of still water quickly forms a region of intense blue coloration around the crystal. If the liquid is left undisturbed for several days, it is found that the coloration will finally extend throughout the whole mass of liquid, becoming diluted to a very pale pink as it does so. This gradual spreading of one liquid (potassium permanganate solution) through another (water) is called *Diffusion*.

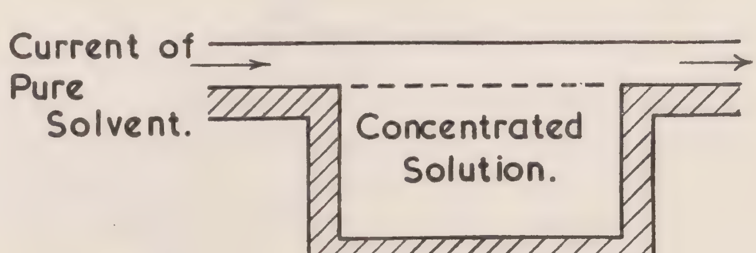


Fig. 9.29

Experiments on rates of diffusion were first performed by Graham in 1851, and experiments, following his method but using the more refined apparatus shown in Fig. 9.29, have elucidated the laws obeyed by the diffusion process.

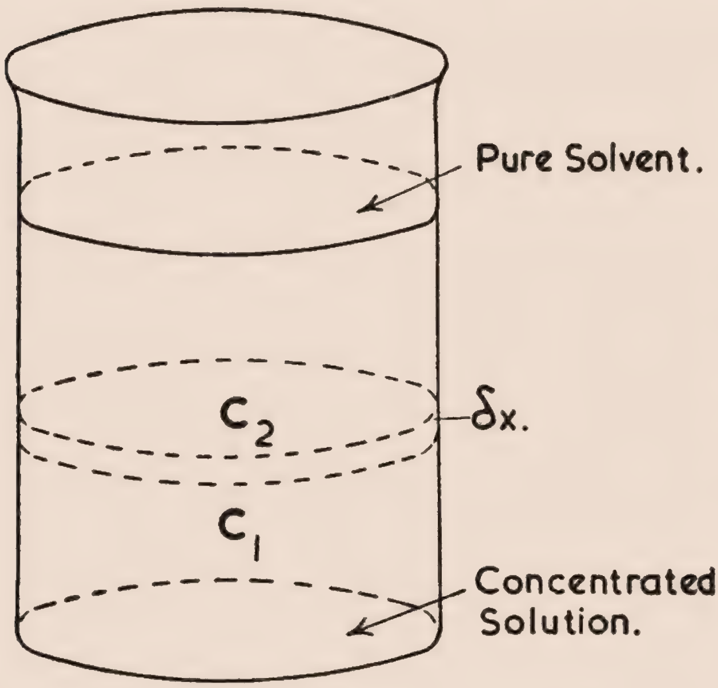


Fig. 9.30

A current of pure solvent flows very slowly at a known rate over the surface of a vessel containing the concentrated solution. Some of the solution diffuses into the solvent; analysis of the solvent, after it has passed over the vessel, enables the amount of solution which has diffused into the solvent to be calculated.

From experiments of this type it has been established that:

(i) The rate at which the solution diffuses into the solvent depends on the solid used to make the solution.

(ii) For a given solution, the rate of diffusion depends on the concentration.

If the diffusion takes place in one direction only (for example, a layer of solution is placed at the bottom of a vertical tube and then covered with a layer of solvent so that diffusion takes place only in the vertical direction), the rate at which the solute diffuses through the solvent is given by an equation developed by Fick and known as his law.

If a slice of the solution of thickness δx is considered, as in Fig. 9.30, and if the concentrations at the top and bottom of the slice are c_2 and c_1 respectively (normally measured in grams of solute per cubic centimetre of solution), then, according to Fick, the mass Q of solute passing through the slice in one second is given by:

$$Q = kA\left(\frac{c_1 - c_2}{\delta x}\right) \quad . \quad . \quad . \quad . \quad (15)$$

where A is the area of cross-section of the tube and k is a constant for the solute known as its '*diffusion constant*' (or sometimes '*diffusivity*').

Many methods have been used to measure the change in concentration of the solution as the diffusion goes on. Perhaps the most intriguing is that adapted by Lord Kelvin, who used a range of beads of different densities floating in the solution. As the diffusion proceeds,

the density of the solution at any point changes due to the change in concentration of solute; each bead sinks to a position where its density is the same as that of the fluid. Thus, by watching the rise or fall of the beads, the changes in concentration of the solution can be followed and its diffusivity measured.

The process of diffusion is accounted for by the small random motions of the molecules of solute and solvent. This causes an intermingling of the two sorts of molecules until one is evenly distributed throughout the other. Although the treatment above has been restricted to liquids, diffusion also occurs in gases and acts very much more rapidly than in liquids owing to the much greater motion of the molecules between collisions.

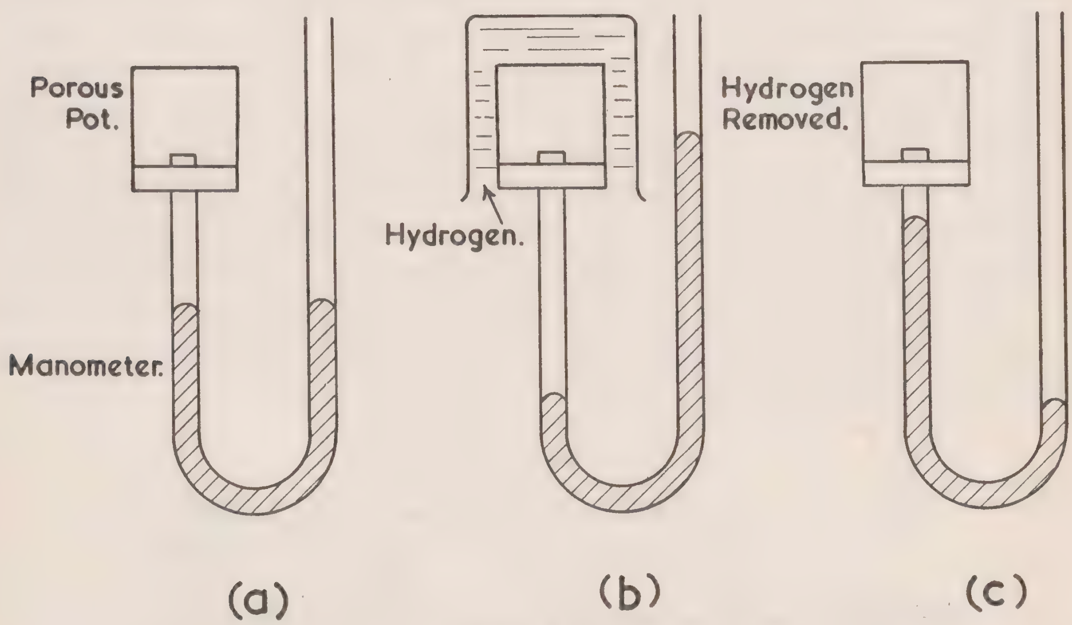


Fig. 9.31

Another process is also described as 'diffusion'. Gas molecules, for example, will pass through the very fine holes in the walls of a porous pot and the gas is said to diffuse through the pot. The effect can be demonstrated with a porous pot arranged as shown in Fig. 9.31 (*a*). If a beaker of hydrogen is inverted over the pot the pressure inside the pot increases as shown by the manometer (Fig. 9.31 (*b*)). If the beaker is now removed, the manometer shows that a partial vacuum occurs inside the pot (Fig. 9.31 (*c*)). This effect can be explained if it is assumed that diffusion takes place in both directions through the pot. In the first case air diffuses outwards as hydrogen diffuses inwards, but the hydrogen diffuses at a greater rate than the air, thus causing pressure inside the pot. In the second case the hydrogen which is now inside the pot diffuses outwards faster than it can be replaced by air and consequently a partial vacuum occurs.

Experiments with various gases have indicated that the rate of

diffusion decreases as the molecular weight of the gas increases; this can be predicted on theoretical grounds from the kinetic theory of gases.

Two cases of the diffusion of gas molecules through a porous barrier must be carefully distinguished. If the tunnels through the barrier have a diameter much larger than the mean free path of the molecules, then as the molecules pass through the holes they will collide with each other much more frequently than they collide with the walls. This is not very different from the condition which obtains in the mass of the gas and thus it flows through the hole as a stream. If the gas contains a mixture of molecules, then all of them, whatever their size, are swept along with the stream, If, however, the tunnels are very small in diameter compared with the mean free path of the molecules, instead of 'flowing' through the hole the molecules will bounce from wall to wall many times whilst getting through. In this case the chance of a molecule passing through the barrier depends very much on the individual molecule and particularly on its molecular weight. If the gas contains a mixture of molecules, some will get through more easily than others, and a mixture of different molecules can be sorted by passing them through such a barrier. The process is called *atmolysis* and has proved very useful for the separation of isotopes, substances which have different atomic weights but the same chemical properties, and hence cannot be separated by chemical means.

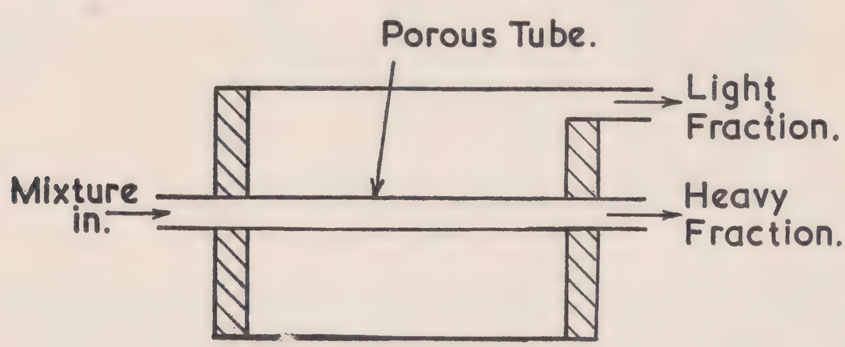


Fig. 9.32

Neon, which is mainly composed of atoms of atomic weights 20 and 22 provides a good example. If such a gas is passed through an apparatus as shown in Fig. 9.32, the lighter isotope of atomic weight 20 diffuses through the walls of the tube at a greater rate than the heavy one and hence the gas is separated into two (rather impure) fractions.

Many repetitions of this process result in almost complete separation of the two isotopes.

9.15 Osmosis

Many forms of membrane are known in which the holes are so small that some molecules cannot pass through them at all. These mem-

branes could be used to sort out a mixture of molecules, but the effect is not very selective and it is usually possible to sort only very big molecules from very small ones. For example, one particular membrane will allow water molecules (molecular weight 18) to pass but rejects sugar molecules of weight 342. Membranes such as this are described as *semi-permeable* and the preferential transfer of one type of molecule is called *Osmosis*.

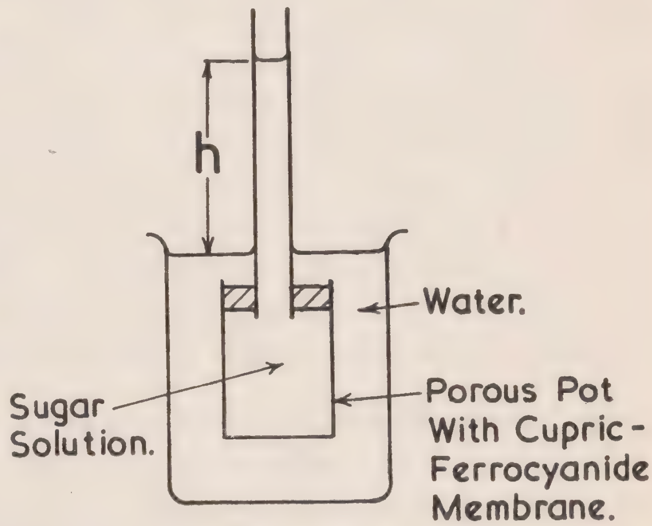


Fig. 9.33

The process can be demonstrated using as a semipermeable membrane a film of cupric ferrocyanide; this is mechanically very weak, but it is formed in the interstices of a porous pot (by interaction between copper sulphate and potassium ferrocyanide) and gains strength thereby. A pot treated in this fashion is filled with sugar solution and then immersed in pure water as shown in Fig. 9.33; very soon the liquid rises in the central tube, showing that water molecules are diffusing into the pot. The rise does not proceed indefinitely, however, but builds up to a definite head h and then stops; the liquid in the pot is then subject to the hydrostatic pressure of the liquid in the tube; this pressure is called the *Osmotic Pressure* of the solution.

This effect is very easily explained if the motion of the molecules is taken into account. Think of the membrane as pierced with a number of holes, rather like a colander; the molecules in the liquid, in a state of continual agitation, bombard the membrane. Every now and then one will score a direct hit on a hole and pass through it, provided that the molecule is small enough. Now if on one side of the membrane there is pure water and on the other a solution of sugar in water, *every* direct hit from the water side will result in the passage of a water molecule through the membrane. Of the direct hits from the solution side, some will be water molecules, which will get through, and some will be sugar molecules, which will not. On the average, therefore, more water molecules will migrate from the pure liquid into the solution than vice versa.

In the demonstration just described, this migration increases the hydrostatic pressure of the solution in the pot. As a result, the rate at which impacts occur on the inside of the membrane increases and so augments the outward flow of water. The pressure continues to grow

until the inward flow is balanced by the outward flow of water; thereafter no further nett increase of material occurs inside the pot and the pressure remains constant.

Osmotic pressure is not merely a hydrostatic phenomenon but is due to the solute molecules; for when equilibrium has been attained, both sides of the membrane are bombarded by the same number of water molecules, hence these cannot cause the pressure difference. The excess pressure on the solution side of the membrane must therefore be due to the bombardment by the sugar molecules. The motion of the solute molecules continues whether or not a semipermeable membrane is present. Thus osmotic pressure is an intrinsic property of a solution; the semipermeable membrane merely offers a convenient method of building up a hydrostatic pressure equal to the osmotic pressure of the solution and so measuring it experimentally.

9.16 Streamline and Turbulent Flow of Fluids

Fluids when in motion normally have two quite distinct modes of transfer. When the velocity is low, the fluid passes an obstacle in what is known rather indiscriminately as 'streamline motion'; if the velocity is much higher, then the motion of the fluid past the obstacle is described as 'turbulent'. The distinction between these two modes is that in streamline motion, 'tubes' can be imagined drawn in the fluid so that if a particle of the fluid starts its journey inside one of these tubes, then it never leaves it throughout its motion (see Fig. 9.34).

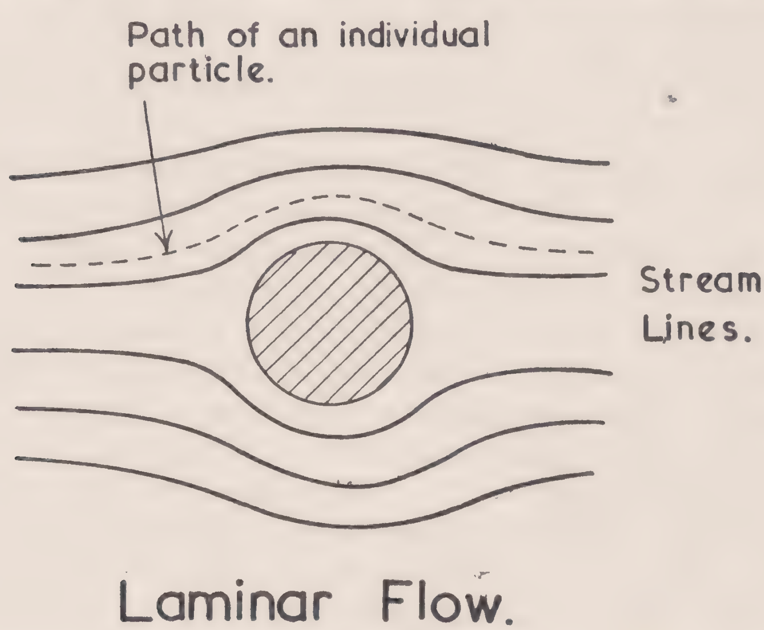


Fig. 9.34

In a one-dimensional flow, the 'tubes' can be replaced by layers of fluid sliding over each other and this has led to the motion being described as *laminar motion*. In turbulent flow, however, the path of a

particle is not confined to a particular tube, but the routes followed by a number of particles cross and intermingle (Fig. 9.35). The change from one type of motion to the other takes place at a critical velocity (depending on the obstacle and the nature of the fluid) and, as might be expected, the two types of motion are governed by very different laws. All of the cases treated below assume that laminar flow is taking place —in general the treatment of turbulent motion is very complex indeed.

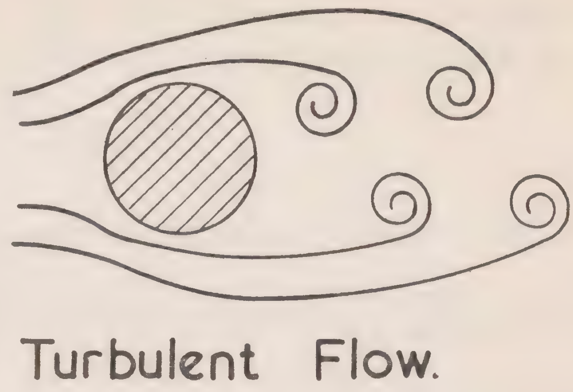


Fig. 9.35

9.17 Viscosity

If a liquid flows in laminar motion over a horizontal plane surface, then each layer of the fluid moves at a different speed; the layer of liquid in contact with the bottom surface remains at rest and the layers move faster as they recede from the bottom.

This effect can be explained by imagining the attractive forces between the molecules of the bottom surface and the lower layer of liquid to be sufficiently strong to hold the liquid molecules at rest against the surface; these molecules then exert a retarding force on the layer of molecules above and so on. A fluid which acts in this way is said to be viscous and possess the property of *viscosity*.

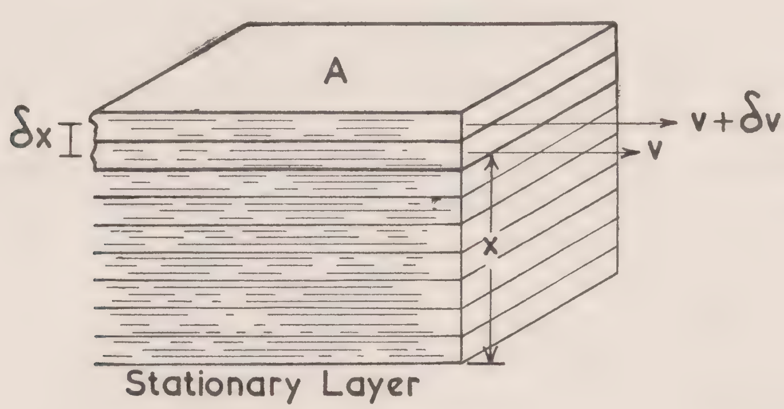


Fig. 9.36

Viscosity is often likened to friction in that a viscous fluid exerts retarding forces on those parts of itself which are trying to move with greater velocity than the rest, just as retarding forces due to friction occur between two solid surfaces in relative motion.

Newton suggested that if a liquid is moving in one dimension with laminar flow, as shown in Fig. 9.36, then the viscous force F exerted on an area A of one layer by the adjacent one is given by:

$$F = \lim_{\delta x \to 0} \eta A \frac{\delta v}{\delta x},$$

where η is some constant depending on the liquid.

Now $\lim_{\delta x \to 0} \frac{\delta v}{\delta x}$ is the velocity gradient $\frac{dv}{dx}$ at the surface between the two layers.

$$\text{Thus } F = \eta A\, dv/dx \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The constant η is called the *coefficient of viscosity* of the liquid.

Re-writing Equation (16) gives:

$$\eta = \frac{F}{A\, dv/dx}.$$

$$\text{Thus } [\eta] = \left[\frac{\text{Force}}{\text{Area} \times \text{Velocity Gradient}}\right]$$

$$= \left[\frac{MLT^{-2}}{L^2 \times LT^{-1}/L}\right]$$

$$= [ML^{-1}T^{-1}].$$

The c.g.s. unit of viscosity is thus the gm.cm^{-1}.sec^{-1}, but this is renamed the 'poise' (after Poiseuille, see page 261).

The f.p.s. unit is the lb.ft^{-1}.sec^{-1}; this is not in common use, the engineer has many arbitrary units in which he measures viscosity.

It is not very easy to verify Equation (16) by direct experiment, but various other expressions (such as that for the flow of a fluid through a tube, see page 261), can be derived from it and have been demonstrated experimentally to be true.

An alternative explanation has been offered for the forces exerted by these layers of liquid on each other. Suppose that a molecule, due to its random motion, migrates from one layer to another moving with a higher velocity. Then in order to pick up the higher velocity the molecule must receive some extra momentum from the other molecules in the layer, i.e. they must exert an accelerating force on it for a short time. The molecule will, however, exert an equal and opposite force on the layer, that is a retarding force. If there is a constant migration of molecules into the layer, the small forces due to the individual molecules will combine into a single steady force.

Similarly, molecules migrating back from the faster into the slower layer will give up momentum and exert an accelerating force on the slower layer.

9.18 Flow through a Tube

When laminar flow takes place through a tube instead of over a flat surface, the laminæ become cylindrical shells concentric with the walls of the tube (Fig. 9.37).

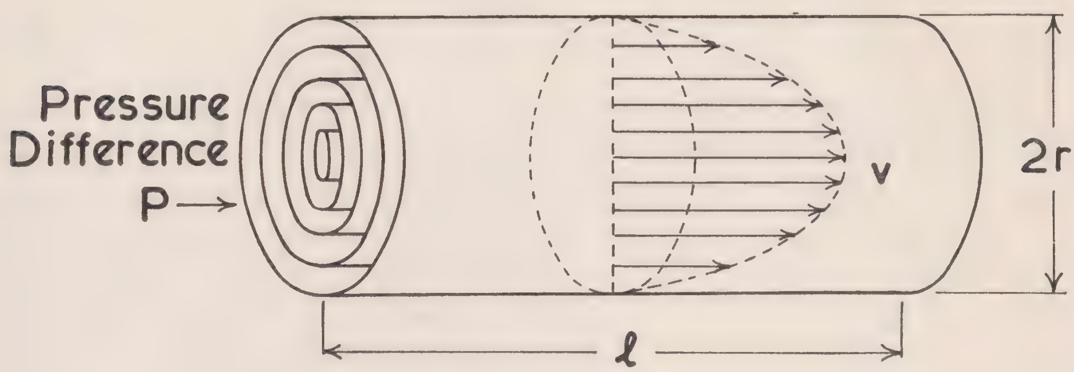


Fig. 9.37

Equation (16) can be extended to cover this case, but its development is rather beyond the scope of this book. It can be shown, however, that the velocity of each layer increases from zero at the wall to a maximum on the axis and follows a parabolic law as shown in the figure, also that the volume of liquid passing through the tube in 1 second is given by:

$$V = \frac{\pi P r^4}{8\eta l} \qquad . \quad . \quad . \quad . \quad (17)$$

where P is the pressure difference between the ends of the tube and η is the coefficient of viscosity of the liquid flowing through the tube. This expression was developed by Poiseuille and provides an easy

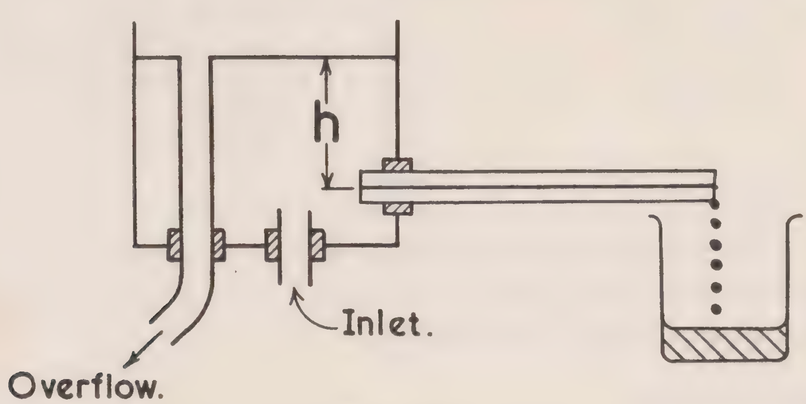


Fig. 9.38

experimental method of measuring viscosity. The liquid is maintained at a constant head in the tank (Fig. 9.38), thus the pressure at the inlet end of the capillary tube is $p_0 + g\varrho h$, whilst that at the outlet end is p_0,

the pressure difference between the ends of the tube is therefore $g\varrho h$ and the volume flowing through the tube in unit time is given by:

$$V = \frac{\pi g \varrho h r^4}{8\eta l}$$

$$\text{or } \eta = \frac{\pi g \varrho h r^4}{8Vl} \qquad . \quad . \quad . \quad . \quad (18)$$

This treatment assumes that the work done by the pressure on the fluid in the tube is all used to overcome the viscous forces; in practice the liquid has to be given some kinetic energy for it to pass through the tube, and thus Equation (18) needs some correction for the highest accuracy, but it is generally sufficient to keep the kinetic energy small by using a very fine capillary tube and a small rate of flow.

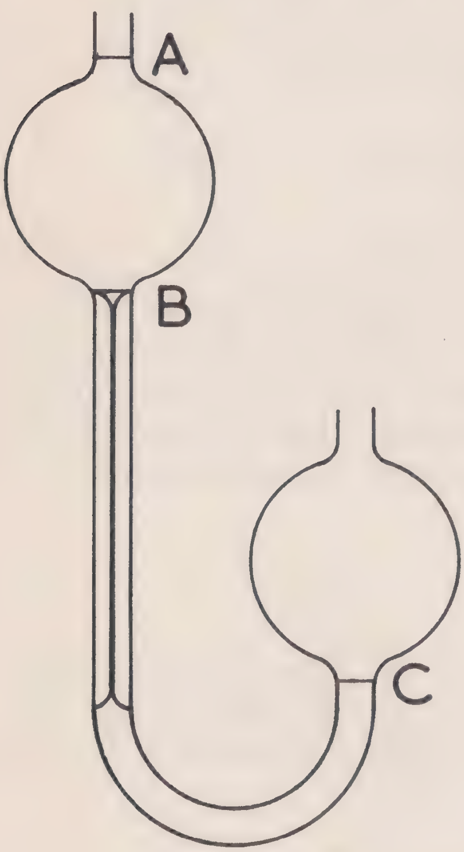


Fig. 9.39

Many industrial viscometers depend on the flow of a liquid through a tube; for example, if the constant head device of Fig. 9.38 were replaced with a simple reservoir, then it would be possible to extend Equation (16) to calculate the time taken for the reservoir to empty itself through the tube and thus a table could be drawn up relating the viscosity of the fluid to the time taken to empty the reservoir.

The standard U-tube viscometer shown in Fig. 9.39 depends on this principle; the tube is filled with oil so that the menisci stand level with the marks A and C. The oil is then allowed to flow from the top reservoir into the lower through the capillary tube until the upper meniscus falls to the level B; the time taken for this to happen is noted and can be converted into a value for the viscosity of the oil, using tables issued by the British Standards Institution.

9.19 Stokes' Law

Stokes, in 1846, extended Newton's law for laminar flow to find the viscous drag on a spherical obstacle when it moves slowly in a straight line and without rotation through a viscous fluid. In this case the force on the sphere is given by:

$$F = 6\pi r v \eta,$$

where r is the radius of the sphere, v its velocity and η the viscosity of the fluid.

Notice that this force is proportional to velocity, so that if a ball is allowed to fall from rest in a viscous fluid under its own weight, the viscous drag is initially zero, and consequently the ball accelerates under its own weight. As its velocity increases, however, so does the viscous drag, until it becomes equal to the weight of the ball. In this event, the resultant force on the ball is zero. Acceleration ceases and the ball continues to 'coast' with a constant velocity; it has reached what is commonly called its *terminal velocity*, symbol v_t.

At its terminal velocity the viscous drag on a sphere is equal to the weight of the sphere when immersed in the fluid (weight of sphere in air less weight of fluid displaced), thus:

$$6\pi r\eta v_t = \tfrac{4}{3}\pi r^3\varrho g - \tfrac{4}{3}\pi r^3\sigma g,$$

where ϱ is the density of the material of the sphere and σ is the density of the fluid. This gives:

$$6\pi r\eta v_t = \tfrac{4}{3}\pi r^3 g(\varrho - \sigma)$$

$$\text{or } \eta = \frac{2}{9}\frac{r^2 g}{v_t}(\varrho - \sigma) \qquad . \quad . \quad . \quad . \quad (19)$$

Thus the viscosity of a fluid may be found merely by measuring the terminal velocity of a sphere falling through it.

Experimentally, some refinements are needed, but for these the reader is referred to *Experimental Physics*, by Daish and Fender, published by the English Universities Press, Ltd.

9.20 Method of Dimensions

Many of the problems in fluid flow cannot be tackled by rigorous methods in our present state of learning, but yield at least a partial solution when attacked using the 'method of dimensions'.

This method depends on the fact that a physical equation must be satisfied not only numerically but dimensionally as well, i.e. the two sides of the equation must have the same dimensions (this was examined on page 4). Provided, then, that the physical quantities which go into an equation are known, the form of the equation can usually be found by arranging the quantities so that the equation is satisfied as far as its dimensions are concerned.

This is best illustrated by an example. Suppose that the hydrostatic pressure produced by a column of liquid is known to depend on the height of the column, the density of the liquid and the acceleration due to gravity, but it is not known how these factors are combined. This statement can be expressed as: the pressure p is some function of h, ϱ, and g, or:

$$p = f(h, \varrho, g) \qquad . \quad . \quad . \quad . \quad (20)$$

If the function assumes a power form, then it may be written as:

$$p = h^x \times \varrho^y \times g^z \quad . \quad . \quad . \quad . \quad (21)$$

where x, y and z are unknown powers.

Writing out the dimensions of the equations gives:

$$[ML^{-1}T^{-2}] = [L]^x \times [ML^{-3}]^y \times [LT^{-2}]^z \quad . \quad . \quad (22)$$

This equation must be satisfied in each dimension, i.e. M must be raised to the same power on each side of the equation, as also must L and T; thus the indices of M, L and T respectively must balance on both sides of the equation. This means that three subsidiary equations can be set up by equating indices for M, L and T, from each side of Equation (22), giving:

$$\text{For } M: \quad 1 = y.$$

$$\text{For } L: \quad -1 = x - 3y + z.$$

$$\text{For } T: \quad -2 = -2z.$$

These three equations can be used to solve for the three unknown powers, x, y and z.

$$\text{Thus } x = 1$$

$$y = 1$$

$$z = 1$$

and substituting these values in Equation (21) gives:

$$p = g\varrho h,$$

which is the same as the expression developed by other means on page 160.

At first sight this method appears to be the panacea for all troubles, for it will provide the answer to all problems merely by the solution of a set of simultaneous equations; actually it is beset with many restrictions which must be carefully examined.

(*a*) If the wrong physical quantities had been included in Equation (20) the method of dimensions would lead to the wrong solution; for example, if the completely fallacious assumption were made that the period of a pendulum depends on Young's Modulus, Y, for the material of the string, the radius, r, of the bob and, ϱ, the density of the surrounding air, then the method of dimensions leads to:

$$T = \sqrt{\frac{\varrho r^2}{Y}},$$

which is dimensionally true but physically nonsensical.

This is the biggest limitation of the method—it can ensure only that the answer is dimensionally correct; the physical truth of the answer has to be ensured by other means. Usually, when examining a new problem by dimensional methods, one's experience of similar problems in physics enables an intelligent guess to be made of the correct physical

quantities to be included, and this guess can always be tested experimentally when the final expression has been obtained. For example, a very easy experiment would show that Young's Modulus for the cord of a simple pendulum does *not* enter into the expression for the period as shown above.

(*b*) Equation (22) merely attempts to balance the dimensions of Equation (21). If this contains a factor with no dimensions, i.e. a pure number, then this factor will not appear in Equation (22) and hence can take no part in the dimensional argument nor appear in the final solution. Thus, if the viscous force F on a sphere of radius r falling through a viscous liquid is assumed to depend on the radius of the sphere, the viscosity of the liquid, and the velocity v of the sphere, this may be expressed as:

$$\begin{aligned} F &= f(r, \eta, v) \\ &= kr^x \eta^y v^z \qquad \text{(23)} \end{aligned}$$

where k is a numerical factor and η is the coefficient of viscosity of the fluid.

$$\text{Thus } [F] = [k]\ [r]^x\ [\eta]^y\ [v]^z$$

$$\text{or } [MLT^{-2}] = [O]\ [L]^x\ [ML^{-1}T^{-1}]^y\ [LT^{-1}]^z.$$

Equating indices gives:

$$\begin{aligned} \text{For } M\text{:}\quad 1 &= y. \\ \text{For } L\text{:}\quad 1 &= x - y + z. \\ \text{For } T\text{:}\quad -2 &= -y - z. \\ \text{Thus } x &= 1 \\ y &= 1 \\ z &= 1, \end{aligned}$$

and substituting these values in Equation (23) gives:

$$F = kr\eta v \qquad \text{(23a)}$$

The method of dimensions cannot give the value of k, and the complete analysis used by Stokes is needed to show that $k = 6\pi$ in this case, and therefore $F = 6\pi r\eta v$.

However, 'knowing the answer' as far as Equation (23*a*) is often a great help in framing the exact analysis of the problem.

(*c*) The next assumption in the method is that the expression is a power function as Equation (23), this is not always true—trigonometric and exponential functions do occur, also added terms of different form but the same dimensions. Thus the velocity of ripples on the surface of a liquid is given by:

$$v = \sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi S}{\lambda\varrho}} \qquad \text{(24)}$$

where λ is the wavelength of the ripples, S the surface tension of the liquid and ϱ its density.

$$\text{Now } \left[\frac{g\gamma}{2\pi}\right] = [LT^{-2} . L]$$
$$= [L^2T^{-2}]$$
$$\text{and } \left[\frac{2\pi S}{\lambda\varrho}\right] = \left[\frac{MT^{-2}}{L . ML^{-3}}\right]$$
$$= [L^2T^{-2}].$$

These two terms are thus of the same dimensions, both equal to the square of a velocity, and they can legitimately be added together. The method of dimensions, if applied to this problem, would correctly indicate both of these terms, but not the fact that they are to be added.

(*d*) A further limitation on the method is that since only three equations can be built up in terms of the indices of Equation (23) (one each for M, L and T), these equations can be solved explicitly only if they contain three variables or less, i.e. functions such as Equation (23) must depend on three physical quantities at the most.

If more than three quantities are present, however, all hope is not lost, as will be seen by working out the case of the ripples quoted above. We have:

$$\begin{aligned} v &= f(g, \lambda, S, \varrho) \\ &= kg^x\lambda^yS^z\varrho^p \quad . \quad . \quad . \quad . \quad . \quad . \quad (25) \\ \text{Thus } [v] &= [g]^x \times [\lambda]^y \times [S]^z \times [\varrho]^p \\ \text{or } [LT^{-1}] &= [LT^{-2}]^x \times [L]^y \times [MT^{-2}]^z \times [ML^{-3}]^p \end{aligned}$$

Equating indices:

$$\begin{aligned} &\text{For } M: & 0 &= z + p. \\ &\text{For } L: & 1 &= x + y - 3p. \\ &\text{For } T: & -1 &= -2x - 2z. \end{aligned}$$

Solving in terms of p gives:

$$\begin{aligned} x &= \tfrac{1}{2} + p \\ y &= \tfrac{1}{2} + 2p \\ z &= -p, \end{aligned}$$

and substituting in Equation (25) gives:

$$v = k\sqrt{g\lambda}\left\{\frac{g\lambda^2\varrho}{S}\right\}^p \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Alternatively, solving in terms of x gives:

$$\begin{aligned} p &= x - \tfrac{1}{2} \\ y &= 2x - \tfrac{1}{2} \\ z &= \tfrac{1}{2} - x, \end{aligned}$$

which, on substituting in Equation (25) leads to:

$$v = k\sqrt{\frac{S}{\lambda\varrho}}\left\{\frac{g\lambda^2\varrho}{S}\right\}^x \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Now the factor $g\lambda^2\rho/S$ which appears in both equations is dimensionless:

$$\left[\frac{g\lambda^2\varrho}{S}\right] = \left[\frac{LT^{-2} \,.\, L^2 \,.\, ML^{-3}}{MT^{-2}}\right]$$
$$= [M^0L^0T^0].$$

The factor is thus a number, indeed it must be if it is to be raised to the arbitrary powers p or x without upsetting the dimensions of the equations. Using this fact, Equations (26) and (27) can be written as:

$$v = k_1\sqrt{\lambda g}$$
$$\text{and } v = k_2\sqrt{\frac{S}{\lambda\varrho}},$$

which reveal the two terms that go to make up the correct Equation (24).

This characteristic of the method is well known, and if it does indicate several answers, then experiments are performed to test whether the correct answer consists of the sum of the alternative solutions.

The difficulty of handling more factors than three may sometimes be removed by grouping two factors into one, thus if it is assumed that the volume of liquid flowing through a tube in unit time depends on the pressure applied to the liquid, the radius and length of the tube and the viscosity of the liquid, then:

$$V = f(p, r, \eta, l)$$

(where V stands for the *volume per second*).

$$\text{Thus: } V = kp^x r^y \eta^z l^q,$$

and we are again faced with four variables. If, however, it can be assumed that p and l are always related in the fashion (p/l), i.e. it is the pressure gradient in the tube that influences the flow, then:

$$V = k(p/l)^x r^y \eta^z \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$\text{Thus: } [V] = [p/l]^x\,[r]^y\,[\eta]^z$$

$$\text{or } [L^3T^{-1}] = \left[\frac{ML^{-1}T^{-2}}{L}\right]^x [L]^y\,[ML^{-1}T^{-1}]^z.$$

Equating indices gives:

For M: $\quad 0 = x + z.$
For L: $\quad 3 = -2x + y - z.$
For T: $\quad -1 = -2x - z.$

This gives:

$$x = 1$$
$$y = 4$$
$$z = -1$$

and substituting these values in Equation (28) gives:

$$V = k \,.\, (p/l)\,\frac{r^4}{\eta}.$$

$$= k \,.\, \frac{pr^4}{\eta l}$$

It was seen in Equation (17), as a result of more complete analysis, that the factor k has the value $\pi/8$.

The foregoing may give the impression that the use of the method of dimensions is hedged around with so many restrictions as to be of no value except in the simplest of cases; this, however, is not true, for with careful handling the method is extremely powerful, and with its aid many problems have been solved which have defied attack by other methods.

9.21 Reynolds' Number

It was mentioned in Section 9.16 that the flow of a fluid past an obstacle changes from streamline to turbulent flow at a critical velocity; it is obviously important to find a value for this velocity, and for this the method of dimensions can be used.

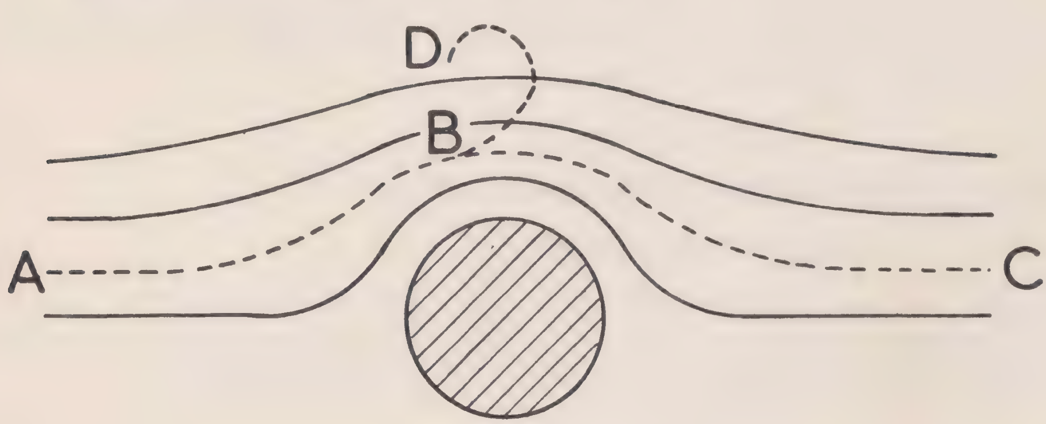


Fig. 9.40

First the factors likely to influence the problem must be discussed. For the motion to be streamlined, a particle must follow the path ABC (Fig. 9.40) and not ABD. This means that the momentum that it picks up at right angles to the general direction of the stream in avoiding the obstacle must not be large enough to overcome the viscous forces set up by the liquid. This momentum will depend on the size of the obstacle, the velocity of the stream and the density of the liquid,

while the viscous forces will also involve the viscosity of the liquid. Thus if v_c is used for the critical velocity at which streamline motion breaks down, then

$$v_c = f(l, \varrho, \eta),$$

where l is some characteristic dimension of the obstacle, and ϱ and η are the density and viscosity of the liquid respectively.

$$\text{Then } v_c = kl^x\varrho^y\eta^z \qquad \ldots \qquad (29)$$

$$\text{or } [v_c] = [l]^x[\varrho]^y[\eta]^z$$

$$\text{thus } [LT^{-1}] = [L]^x[ML^{-3}]^y[ML^{-1}T^{-1}]^z.$$

Equating indices gives:

$$\begin{aligned} \text{For } M\text{:} \quad 0 &= y + z. \\ \text{For } L\text{:} \quad 1 &= x - 3y - z. \\ \text{For } T\text{:} \quad -1 &= -z. \\ \text{Hence } x &= -1 \\ y &= -1 \\ z &= 1, \end{aligned}$$

and substituting these values in Equation (29) gives:

$$v_c = \frac{k\eta}{l\varrho}$$

$$\text{or } k = \frac{v_c l\varrho}{\eta} \qquad \ldots \qquad (30)$$

where k is the unknown numerical constant.

This expression was derived by Reynolds and he showed, by experiment, that k takes the approximate value 1150 at the critical velocity, for all liquids and for all types of obstacle of a particular form. The critical number changes to a new value if objects of a different shape are chosen. To ensure, therefore, that an experiment is being carried out under streamlined conditions, the value of the critical velocity can be found from:

$$v_c = \frac{1150\eta}{l\varrho}$$

and in the experiment the velocity of the fluid must not approach this critical value.

In the two fluid-flow methods discussed in this chapter—flow through a tube and the falling-ball method—l (size of obstacle) can be interpreted as the radius of the tube in the first case and v_c gives the critical velocity of flow through the tube. The velocity calculated from the rate of flow through the tube gives only the average velocity of the liquid in the tube (it increases from zero at the wall to a maximum on the axis) and so must be well below the critical velocity if this is not to be exceeded at the axis. In the case of the falling ball, the size of the

obstacle, l, is the radius of the ball and v_c the largest velocity with which it can fall if the liquid is to pass around it in streamline fashion.

Equation (30) is, however, used in another way. It will be seen that if l, η, and ϱ are constant (as they usually are in one experiment), then k is proportional to the actual velocity v of the fluid in the apparatus, and takes the specific value 1150 only when v is equal to v_c.

This may be written as:

$$\frac{k}{1150} = \frac{v}{v_c}.$$

For streamlined motion, v must be much less than v_c, and thus the value of k obtained for the actual flow through the apparatus must be much less than 1150. When used in this sense, k is called *Reynolds' Number*, and 1150 is the critical value of Reynolds' Number.

If the value of Reynolds' Number for the conditions obtaining in an actual experiment is found to be 120, the motion will be very stably streamlined, if it is 1000 the motion is getting dangerously close to the critical region where it can turn from streamline to turbulent motion, and if it is 2500, then the motion will undoubtedly be turbulent.

SUMMARY OF NEW UNITS INTRODUCED IN THIS CHAPTER

Quantity	*c.g.s. unit*	*f.p.s. unit*	*M.K.S. unit*	*Gravitational unit*
Surface Energy .	$erg.cm^{-2}$	Not in common use	$joule.m^{-2}$	Not in common use
Surface Tension	$dyne.cm^{-1}$	$poundal.ft^{-1}$	$newton.m^{-1}$	$lb\text{-}wt.ft^{-1}$
Viscosity . .	$gm.cm^{-1}.sec^{-1}$ or poise	$lb.ft^{-1}.sec^{-1}$	$kg.m^{-1}.sec^{-1}$	—
Reynolds' Number	Pure Number			

EXERCISES 9

1. Describe the phenomenon of *surface tension*, mentioning three illustrative examples. How may the existence of surface tension be explained?

   Calculate the rise of a liquid B in a capillary tube of radius 0·13 mm. if that of liquid A is 2·0 cm. in a tube of radius 0·26 mm. The surface tension and the density of liquid A are respectively 0·6 and 1·2 times the corresponding quantities for liquid B. Prove any formula employed in the calculation.

   (London Univ. G.C.E. Advanced level.)

2. Describe and give the theory of the capillary-rise method of measuring the surface tension of water at room temperature.

Water, surface tension 72 dyne.cm.$^{-1}$, rises to a height of 20 cm. in a vertical capillary tube of uniform bore. What is the radius of the tube?

The above tube is removed from the water, dried and its upper end sealed. The tube, of total length 40 cm., is then held vertically with its lower end just touching a water surface. If the atmospheric pressure is equal to that of a vertical column of water 10 metres high, to what height will the water rise in the tube? (An approximate solution will be accepted.) (London Univ. Inter. B.Sc.)

3. Describe and explain an experimental method for the determination of the ratio of the values of the surface tension of water and methylated spirit, assuming that the angle of contact of each against glass is zero.

   A glass capillary tube having a uniform internal diameter of 1·00 mm. is placed vertically with one end dipping into water, the surface tension of which is 70 dyne.cm.$^{-1}$. Calculate the height to which the meniscus rises in the tube.

   What will happen if the tube is lowered until only 2 cm. of its length is out of the water? (Northern Univ. H.S.C.)

4. Describe a method of measuring the surface tension of water.

   A common hydrometer is made with a cylindrical stem of radius 1·5 mm. and floats upright in alcohol which wets the stem. The density of alcohol is 0·785 gm./c.c. and the surface tension is 25·6 dynes/cm. Calculate how much the hydrometer would rise if the surface tension of alcohol were zero.

   (London Univ. G.C.E. Advanced level.)

5. Define *surface tension* and *angle of contact.*

   Derive an expression for the rise of a liquid in a capillary tube, and describe how you would measure the surface tension of water by this method.

   A uniform capillary tube, open at both ends, is 0·5 mm. in internal diameter. It is lowered vertically into mercury and held with part of the tube protruding from the mercury. Find the difference in level of the mercury inside and outside the tube. The surface tension of mercury is 470 dynes per cm., the angle of contact with glass is 130°, and the density of mercury is 13·6 gm. per c.c.

   (Oxford G.C.E. Advanced level.)

6. Define *surface tension, angle of contact.*

   Describe how you would measure the surface tension of a liquid which wets glass, and give the theory of your method.

   Estimate the smallest diameter which the upper part of a barometer tube may have in order that the capillarity error shall be less than one millimetre. (Take the density of mercury to be 13·6 gm. per c.c., the surface tension of mercury to be 450 dynes per cm., and the angle of contact with glass to be 126°.) (Oxford G.C.E. Advanced level.)

7. Explain what is meant by surface tension and describe *two* experiments to illustrate it.

   Why does a liquid rise in a capillary tube? How high will an oil rise

in a tube of 0·4 mm. diameter if the oil has a density of 0·85 gm./cm.^{3} and a surface tension of 26 dynes/cm. and if the angle of contact is 30°? (Cambridge H.S.C.)

8. What is meant by surface tension? Give *three* examples of simple observations which demonstrate its existence.

The lower end of a vertical tube 2 mm. in diameter dips into soap solution, and on the upper end is a soap bubble 20 mm. in diameter. How much is the level of the liquid in the tube above that of the surrounding soap solution? (Surface tension of soap solution 27 dynes/cm.; the density of soap solution may be assumed to be 1.)

(Cambridge H.S.C.)

9. It is proposed to determine the surface tension of water by a capillary tube method.

(*a*) Discuss the considerations that influence the choice of a suitable tube.

(*b*) State the precautions taken in preparing the apparatus.

(*c*) Describe in detail how you would measure height and radius.

(*d*) Prove the formula used to calculate the result.

A bubble, 1 cm. radius of soap solution (surface tension 26 dyne.cm.$^{-1}$) is blown on the end of a glass tube joined to a U-tube manometer containing water. What is the difference of levels of the water in the manometer? Comment on the suitability of this method when used to determine the surface tension of soap solution.

(Cambridge G.C.E. Advanced level.)

10. Define *surface tension*. Describe and explain two experiments which illustrate its effects.

Derive an expression for the pressure difference between the inside and outside of a spherical soap bubble, and hence calculate the pressure inside a small air bubble of 0·014 cm. radius situated just below the level of a water surface exposed to an atmospheric pressure of 76·00 cm. of mercury. (Surface tension of water = 70·00 dynes per cm.)

(London Univ. G.C.E. Advanced level.)

11. Two vertical parallel sheets of glass separated by a distance 0·2 mm. dip into clean water. The liquid rises between the plates to a height of 7 cm. What is the surface tension of the water? If the plates are 10 cm. broad, estimate the forces required to maintain the given separation. (Manchester Univ. Schol.)

12. Describe and explain two different methods, involving the use of a glass tube of fine bore, of measuring the surface tension of a liquid.

Two circular glass plates, each 5 cm. radius, are separated by a film of water 0·01 mm. thick. Find in kgm.wt. the minimum direct pull required to separate them. Give the theory underlying the calculation. The surface tension of water is 73·0 dynes per cm.

(Northern Univ. G.C.E. Schol. level.)

13. A glass capillary tube is 10 cm. long and has internal and external diameters of 2 mm. and 4 mm. respectively. Its mass in air is 5·76 gm. and, when completely immersed in a given liquid which wets glass,

its apparent mass is 4·51 gm. When suspended with its axis vertical and its lower end 5 cm. below the free surface of the liquid, its apparent mass is 5·26 gm. Find the surface tension and calculate how far the liquid inside the capillary has risen above the free surface. ('g' $= 981$ cm.sec.$^{-2}$) (Oxford Univ. Schol.)

14. The excess pressure inside a bubble of radius 1 cm. formed from a certain liquid maintains a difference of 2 mm. between the oil levels in a U-tube manometer. If a capillary tube of radius 1 mm. is dipped into a quantity of the liquid, the liquid column rises to a height of 6 mm. above the free surface. Derive formulæ which show how these effects can be explained in terms of a surface tension, and point out the further measurements you would make in order to find its value. What conclusion can you draw if you are told that the liquid and oil densities are 1·2 gm.cm.$^{-3}$ and 0·8 gm.cm.$^{-3}$ respectively? (Cambridge Univ. Schol., King's College Group.)

15. Enumerate *four* phenomena which demonstrate surface tension, explaining carefully how the phenomena you choose depend on surface tension.

A drop of water of volume 1 cm.^{3} is placed on a glass plate. A second glass plate is placed over it and it is observed that the water has spread over a circular area of radius 5 cm. Calculate the force between the two plates.

The plates are now pressed together so that the distance separating them is halved. How does the force between the plates change? (Surface tension of water is 73·5 dyne/cm.)

(Cambridge Univ. Schol., Girton and Newnham Colleges.)

16. How would you determine the surface tension of water as a function of the temperature?

A spherical drop of liquid, after being subjected to a small deformation, executes small oscillations about its equilibrium shape. Find by the method of dimensions how the frequency of oscillation will depend on the surface tension, the density of the drop and its radius. (Oxford Univ. Schol.)

17. Define *coefficient of diffusion*. State the laws of diffusion of salts in dilute aqueous solutions.

Explain how the osmotic pressure of a dilute aqueous sugar solution may be determined. (London Univ. Inter. B.Sc.)

18. Describe an experiment to show how the volume of water flowing in unit time through a narrow tube depends on the pressure difference between the ends of the tube. Explain what happens when the pressure difference between the ends is progressively increased to a high value.

Water from a reservoir flows through a capillary tube and the level in the reservoir falls from a mark A to a mark B in 2 min. When another liquid, of specific gravity 0·96, is substituted for water the level falls from A to B in 100 sec. Calculate the ratio of the viscosity of the liquid to that of water.

(Northern Univ. G.C.E. Advanced level.)

19. Distinguish between *orderly* and *turbulent* motion of a fluid and describe an experiment to illustrate the difference. Discuss the advantage gained by stream-lining racing cars.

In an experiment to determine the viscosity of water at room temperature it was observed that a pressure difference of 5·6 cm. of water between the ends of a horizontal capillary tube, 10·0 cm. long and 0·50 mm. internal radius, caused water to emerge at the rate of 8·1 c.c. per minute. Draw a diagram to show the arrangement of apparatus suitable for performing the experiment and calculate a value for the viscosity of water, assuming that the rate of flow through the tube is equal to $\pi pa^4/8\eta l$ where the symbols have their usual meaning. (Northern Univ. G.C.E. Advanced level.)

20. Define *coefficient of viscosity*. What are its dimensions?

By the method of dimensions, deduce how the rate of flow of a viscous liquid through a narrow tube depends upon the viscosity, the radius of the tube, and the pressure difference per unit length. Explain briefly how you would use your results to compare the coefficients of viscosity of glycerine and water.

(Cambridge H.S.C.)

21. Explain the method of dimensions for finding the relation between physical quantities, and give *two* examples to illustrate both its use and its limitations.

On the assumption that the flow of liquid in a tube becomes turbulent at some critical velocity V which depends only on the viscosity η, the density of the liquid ρ and the radius of the tube r, find how V varies with these other quantities.

(Cambridge G.C.E. Schol. level.)

22. Define *coefficient of viscosity*. Distinguish between orderly and turbulent flow of a liquid through a tube. Describe a method to determine for a given tube and liquid the pressure head at which the transition from orderly to turbulent flow occurs.

A horizontal capillary tube, 50 cm. long and 0·20 mm. internal radius, is inserted into the lower end of a tall cylindrical vessel of cross-sectional area 10 sq. cm. The vessel is filled with water which is allowed to flow out through the tube. Calculate the time taken for the level of the water in the vessel to fall from a height of 100 cm. to 50 cm. above the axis of the tube. Assume that the volume of water passing per second through a horizontal tube is $\pi a^4(p_1 - p_2)/l8\eta$, where a = tube radius, l = tube length, η = coefficient of viscosity of water and $(p_1 - p_2)$ = difference in the pressures at the ends of the tube. Take the viscosity of water as 0·010 gm.cm.$^{-1}$sec.$^{-1}$ and $\log_e 10 = 2·30$. (Northern Univ. G.C.E. Schol. level.)

23. Explain what is meant by *viscosity* and define *coefficient of viscosity*.

According to Stokes's law the motion of a sphere of radius a having a small velocity v in a medium of viscosity η is opposed by a force $6\pi\eta av$. Using this information, explain why a small sphere falling through a viscous medium acquires a terminal velocity.

In an experiment to determine the viscosity of glycerine a steel

ball of diameter 3 mm. falling through glycerine contained in a vertical tube is observed to fall with constant velocity through 25 cm. in 7·5 seconds. If the densities of glycerine and steel are 1·3 and 7·7 gm. per c.c. respectively, calculate the coefficient of viscosity of glycerine at the temperature of the experiment and state the assumptions which you make. How will the value of the coefficient depend on the temperature? (Northern Univ. G.C.E. Advanced level.)

24. Under suitable conditions the force F opposing the motion of a sphere through a liquid depends on the radius a of the sphere, its velocity v, and the coefficient of viscosity η of the fluid. Use the method of dimensions to find the form of the expression relating F, a, η and v.

Taking the densities of lead and of glycerine to be 11·3 and 1·3 gm.cm.$^{-3}$ respectively, and the viscosity of glycerine to be 12·1 c.g.s. units, find the value of the numerical constant involved in this expression for F, given that the terminal velocity of a lead sphere 1 mm. in diameter falling vertically through glycerine is 0·45 cm. sec.$^{-1}$. (Oxford H.S.C.)

25. Explain the 'method of dimensions'.

Show that the viscous force on a sphere of radius r moving with velocity v through a fluid is equal to $\eta rv \times$ a constant, where η is the coefficient of viscosity of the fluid. If the constant is 6π, find the limiting velocity attained by a ball of diameter 3 cm. and mass 1 gm. released from the bottom of a deep tank of water. (Coefficient of viscosity of water = 0·1 c.g.s. units.) (Oxford Univ. Schol.)

26. A small sphere, allowed to fall from rest in a viscous medium, acquires a terminal velocity. Account for this qualitatively and discuss the various factors which determine the magnitude of the velocity.

Describe how you would compare the viscosities of two liquids by measurements of terminal velocities.

Calculate the radius of a mercury droplet which, falling through oil of specific gravity 0·92, acquires the same terminal velocity as an air bubble of radius 0·10 mm. rising through water. (Assume that the specific gravities of mercury and air are 13·6 and 0·0013 respectively and that the viscosity of the oil is 80 times that of water.) (Northern Univ. G.C.E. Schol. level.)

27. State the laws governing solid and fluid friction and indicate how they differ from one another.

A steel ball-bearing of radius r cm. and density ρ gm./cm.^{3} is released from rest just under the surface of a liquid of density ρ_0 gm./cm.^{3} at rest within a large tank. What is the *initial* downward force in dynes acting on the ball-bearing? If the resisting force at any instant experienced by a sphere in its orderly motion through an infinite liquid medium is given by $R = 6\pi r\eta v$, where η is the viscosity of the liquid and v is the instantaneous velocity of the sphere, derive an expression for the constant *velocity* attained by the ball-bearing. How may this result be applied to the measurement of the viscosity of liquids?

Use the following data to calculate the terminal velocity of a steel ball-bearing falling freely through a certain liquid:

Mass of steel ball-bearing = 0·0352 gm.
Density of steel = 8·4 gm./cm.^{3}
Density of liquid = 1·3 gm./cm.^{3}
Viscosity of liquid = 8·0 gm./cm./sec.

(London Univ. G.C.E. Schol. level.)

28. Define the coefficient of viscosity of a fluid.

A thin wooden disc of mass 300 gm. and diameter 20 cm. floats on a large water surface, with a clearance of 1 mm. between its lower surface and the flat bottom of the container. If the disc is initially rotating about its axis with angular velocity ω, calculate the time for the angular velocity to be reduced to $0{\cdot}1\omega$. (The coefficient of viscosity of water is 0·01 c.g.s. units.) (Oxford Univ. Schol.)

29. Explain the meaning of *terminal velocity* as applied to the motion of a small sphere falling through a viscous liquid.

Describe an experiment to show how the terminal velocity of a ball-bearing falling under gravity in a viscous oil varies with its diameter.

An oil drop carrying a charge of 144×10^{-10} e.s.u. is balanced in air by an electric field of 5,000 volt. cm.$^{-1}$. Determine (i) the radius of the drop, (ii) the terminal velocity acquired after removal of the field. (The resistance experienced by a sphere of radius a moving with velocity v in a medium of viscosity η is given by $6\pi a\eta v$. The densities of the oil and of air are 0·9200 and 0·0013 gm.cm.$^{-3}$ respectively. 300 volt = 1 e.s.u. of potential. Viscosity of air, $1{\cdot}824 \times 10^{-4}$ gm.cm.$^{-1}$sec.$^{-1}$) (Northern Univ. G.C.E. Schol. level.)

30. Define coefficient of viscosity and describe how this quantity may be measured in the case of a liquid.

The coefficient of viscosity, η, for a liquid is measured at 2° C., 5° C., 10° C., 30° C., and 50° C., and found to be 2·60, 1·77, 1·31, 0·80, 0·64 c.g.s. units respectively. Assuming that η varies with temperature T measured in degrees Centigrade according to the relation $\eta = AT^n$, evaluate (graphically or otherwise) values for A and n. (Oxford Univ. Schol.)

31. Prove that the pressure within a spherical soap bubble of radius r and surface tension T exceeds the external pressure by $4T/r$. What would be the corresponding expression if the surface of the soap film were a cylinder of radius r?

Oil is pumped along a steel pipe of internal diameter 10·0 cm. and external diameter 10·4 cm. at a rate which produces a pressure gradient along the pipe of 10^7 dynes/cm.^{2} per kilometre. What is the maximum distance which can be allowed between pumping stations if it is desired that the pressure in the pipe shall never exceed one-tenth of the bursting pressure? (Tensile strength of steel = $2{\cdot}3 \times 10^9$ dynes/cm.^{2}) (Oxford Univ. Schol.)

CHAPTER 10

TRAVELLING WAVES, NATURE AND PROPAGATION OF SOUND

10.1 General Characteristics of Travelling Waves

The sight of waves moving over the surface of water is a very familiar one, they are probably the first waves of which we have any real experience. In this chapter, however, the term 'wave' will be extended to mean any repetitive disturbance that travels through a medium. Other types of wave motion are the 'waggles' which can be made to run along a taut cord, and the pressure variations which travel through the air and which we call sound waves.

By considering waves on water some useful ideas can be gathered concerning general wave motions. The wave is not made up of a heap of water which rushes along the surface. Instead, the water at any point rises up above the undisturbed level and then sinks down again, at the same time water at an adjacent point rises up. This gives the

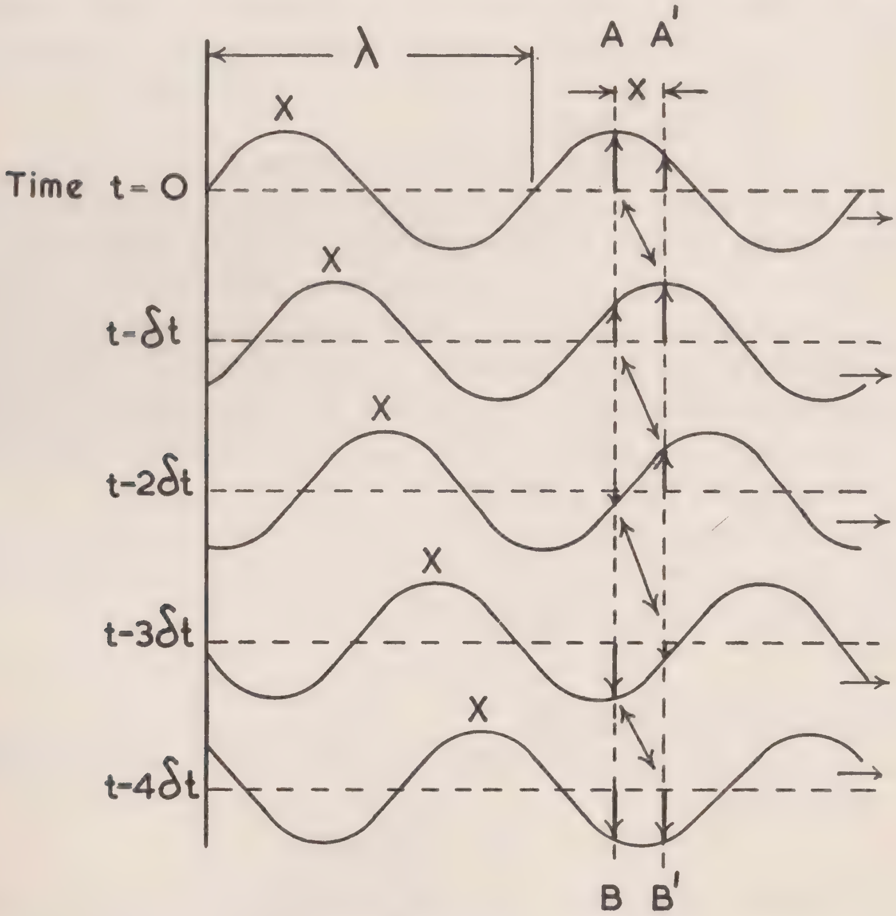


Fig. 10.1

impression that the water is moving along, whereas it is merely moving up and down: things floating on water ride up and over waves but are very seldom swept along by them.

If a series of instantaneous photographs is taken of waves travelling on water, then the displacement of the surface would be as shown in Fig. 10.1. The waves are travelling to the right, as will be seen if the crest X is followed down the diagram, but at a fixed point (running the eye downwards from A to B) the surface motion is merely up and down. This motion of one point in the surface is called the *Particle Displacement,* it is normally a vertical oscillation.

It is typical of all wave motions that there is no general drift of the medium as the wave motion passes through it. But to produce a wave motion a certain amount of work has to be done, for the particles of water acquire both potential and kinetic energy. This work can be recovered at another place by using the up-and-down motion of the water to work a machine. Thus, although a wave motion results in no general transfer of matter, it can be used to transfer energy from one place to another. The *rate* at which this transfer takes place is used to measure the *intensity* of the wave motion.

At this stage, the use of water waves as an analogy must be abandoned, for they are actually more complex than any waves subsequently discussed in this book. The theoretical treatment which follows is applicable to any disturbance whose passage through a medium can be illustrated by a graph such as Fig. 10.1 and leads to a general equation for a travelling wave.

It has been observed that at a point in the medium such as AB (Fig. 10.1), the particle displacement goes through a complete cycle of values. Now if the particle displacement at another point such as $A'B'$ is examined, we see that the displacement at any instant is equal to the displacement that occurred at AB at the previous instant. Thus the cycle of motion at $A'B'$ is the same as that at AB but is performed at a slightly later time. Hence if we could write down the equation representing the motion of the particle at AB, then we should be able to derive from it the equation for the motion at $A'B'$.

The particle displacement can be any motion we please, the one most often encountered is a simple harmonic motion, thus the particle displacement ξ at A can be written as:

$$\xi = a \sin \omega t \quad . \quad . \quad . \quad . \quad (1)$$

(See Equation (65), page 92).

Since $\omega = 2\pi/\tau$ (see Equation (63), page 92), where τ is the periodic time of the motion,

$$\xi = a \sin 2\pi t/\tau \quad . \quad . \quad . \quad (2)$$

In this equation it will be seen that the amplitude of the particle displacement has the value a. The periodic time τ is the time taken

for the particle to make one complete excursion up, down and back again to its starting position—in Fig. 10.1 τ is roughly equal to $8\delta t$.

At $A'B'$, a particle performs the same motion as at AB, but at a time δt later, i.e. the motion at $A'B'$ lags on that at AB by a fraction $\delta t/\tau$ of a cycle, thus it has a phase lag of $2\pi\delta t/\tau$ radians (see page 94) and the motion at $A'B'$ can be written as:

$$\xi' = a \sin\left(2\pi\frac{t}{\tau} - 2\pi\frac{\delta t}{\tau}\right)$$

$$= a \sin\frac{2\pi}{\tau}(t - \delta t).$$

The velocity v of the wave motion can now be introduced as follows.

If x is the distance between A and A' and δt is the time taken by the same point in the wave motion to move on from A to A', then the velocity of propagation of the wave motion is given by

$$v = x/\delta t \qquad (3)$$

$$\text{or } \delta t = x/v.$$

$$\text{Thus } \xi' = a \sin\frac{2\pi}{\tau}\left(t - \frac{x}{v}\right) \qquad (4)$$

This is the general equation of a travelling wave motion, but it may be presented in several alternative forms as shown below.

A particle makes one complete cycle of its motion in a time τ, but the wave motion progresses with a velocity v, and thus in a time interval equal to τ, moves on a distance $v\tau$. Hence in the wave motion there will be two points, separated by a distance $v\tau$, having the same value of particle displacement—the crests of two adjacent waves for example. The distance between two such points is called the *Wavelength*, λ, of the wave motion, thus

$$\lambda = v\tau \qquad (5)$$

Notice that $1/\tau$ is equal to f, the frequency of the particle displacement,

$$\text{hence } v = f\lambda \qquad (6)$$

Substituting for v from Equation (5) into Equation (4) gives:

$$\xi' = a \sin 2\pi\left(\frac{t}{\tau} - \frac{x}{\lambda}\right) \qquad (7)$$

or substituting for τ from Equation (5) into Equation (7) leads to:

$$\xi' = a \sin\frac{2\pi}{\lambda}(vt - x) \qquad (8)$$

Equations (4), (7) and (8) are merely different statements of the same facts. We need therefore examine only one of them in detail, and for this we choose Equation (7).

This equation has been developed to represent the particle motion at a distance x from some starting position, but by giving x all possible values, the equation can be made to represent the motion of every particle making up the wave; even more than this, Equation (7) represents the complete wave motion, as is demonstrated below.

Suppose we concentrate our attention on one fixed position, x then takes a constant value, X say, giving:

$$\xi' = a \sin 2\pi\left(\frac{t}{\tau} - \frac{X}{\lambda}\right).$$

But X is a constant, thus, writing $\frac{2\pi X}{\lambda}$ as some constant ϕ, and remembering that $\frac{2\pi}{t} = \omega$, we have:

$$\xi' = a \sin (\omega t - \varphi),$$

which indicates that the particle displacement at this particular point is a simple harmonic motion.

Next consider the displacement of all particles at a fixed time, i.e. take a snapshot of the displacement when $t = T$.

$$\text{Then } \xi' = a \sin 2\pi\left(\frac{T}{\tau} - \frac{x}{\lambda}\right).$$

But $\frac{T}{\tau}$ is a constant, thus writing $\frac{2\pi T}{\tau}$ as another constant ϕ' gives

$$\xi' = a \sin\left\{\frac{2\pi x}{\lambda} + (\phi' + \pi)\right\} \quad . \quad . \quad (9)$$

(the extra constant π is introduced to avoid the negative sign).

This equation gives the displacement of all particles (by allowing x to take all possible values) at a fixed instant of time. Thus if we run our eye along the snapshot of the motion so that x steadily increases, we see that the profile of the particle displacement is a sine wave of amplitude a. This curve repeats itself whenever the sine term of Equation (9) increases by 2π, i.e. whenever $2\pi x/\lambda$ increases by 2π, 4π, 6π, etc., or x increases by λ, 2λ, 3λ, etc. Thus the wavelength of the motion is λ.

Finally let us follow with our eye some point of constant displacement, say the crest of a wave.

$$\text{Then } \xi' = a$$

$$\text{hence } a = a \sin 2\pi\left(\frac{t}{\tau} - \frac{x}{\lambda}\right).$$

$$\text{Thus} \quad \sin 2\pi\left(\frac{t}{\tau} - \frac{x}{\lambda}\right) = 1,$$

which has as a solution:

$$2\pi\left(\frac{t}{\tau}-\frac{x}{\lambda}\right)=\frac{\pi}{2}$$

$$\text{or}\left(\frac{t}{\tau}-\frac{x}{\lambda}\right)=\frac{1}{4}.$$

Thus $\left(\frac{t}{\tau}-\frac{x}{\lambda}\right)$ is a constant, so if time increases, making the term $\frac{t}{\tau}$ larger, the term $\frac{x}{\lambda}$ must increase by the same amount to keep the expression in the bracket constant. This means that x itself increases and thus, if we wish to keep our eye on one particular crest as time increases, we must follow the crest to increasing values of x, i.e. the wave motion is *travelling* along.

If the increase in time is δt and the corresponding increase in distance is δx, then

$$\frac{\delta t}{\tau}=\frac{\delta x}{\lambda},$$

and the velocity of the motion $\delta x/\delta t$ is given by

$$\frac{\delta x}{\delta t}=\frac{\lambda}{\tau}$$

$$\text{or } v=\frac{\lambda}{\tau},$$

as was seen in Equation (5).

Equations (4), (7) and (8) therefore all represent travelling wave motions which move in the positive direction of x.

A wave motion going in the opposite direction would be written as

$$\xi=a\sin 2\pi\left(\frac{t}{\tau}+\frac{x}{\lambda}\right) \quad . \quad . \quad . \quad . \quad (10)$$

(compare this with Equation (7)).

10.2 Transverse and Longitudinal Waves

So far we have considered a general wave motion, the only stipulation being that its progress could be represented by a graph such as Fig. 10.1. It must now be shown that the waves in which we are interested—waves in stretched strings and sound waves—can be so represented. In all the wave motions considered hereafter the particles of the medium either move to and fro in the same direction as that in which the wave motion progresses or else they move at right angles to this direction.

As examples of these two types of motion consider first the waves which can be made to run along a stretched elastic string if one end is

shaken up and down. Each particle of the string then moves only in a direction *perpendicular* to the general direction of the string, a point at

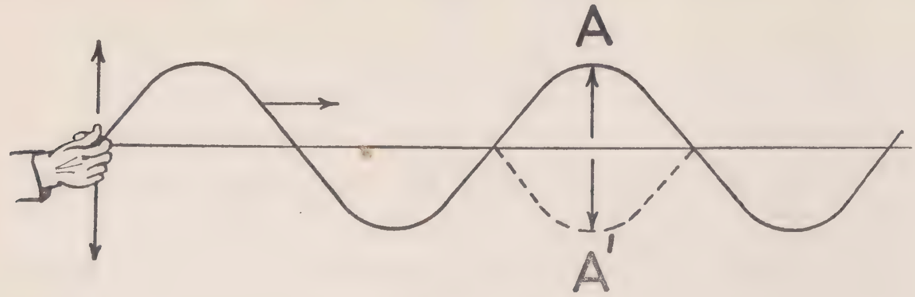


Fig. 10.2

A (Fig. 10.2) moving to A' and back again, while the wave motion travels *along* the string.

These waves are called *Transverse Waves* since the particle displacement is across the direction of travel of the waves.

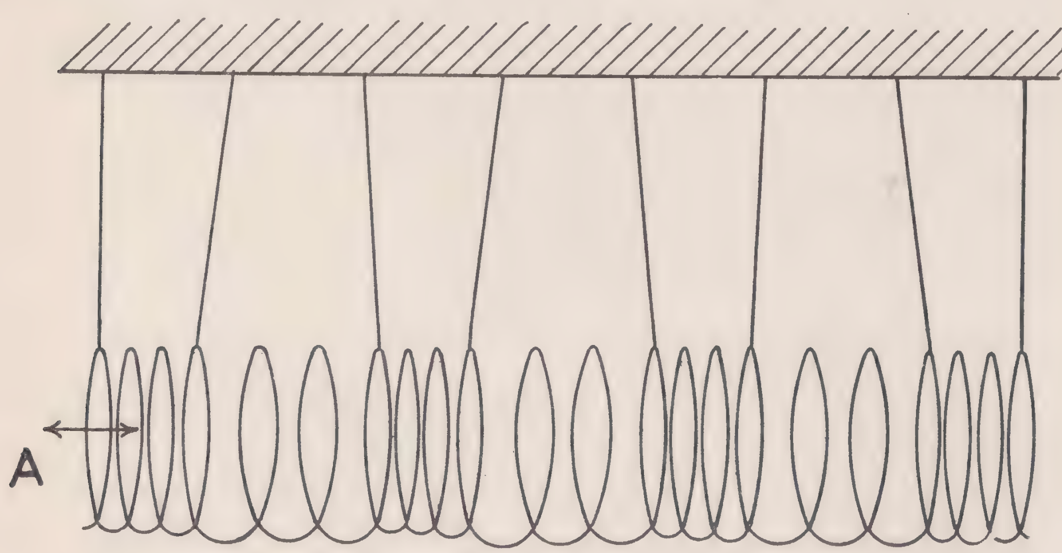


Fig. 10.3

On the other hand, if a coil spring is suspended as in Fig. 10.3 and a back-and-forth motion given to the coils A at one end, a wave motion is seen to run along the spring, but this time the displacement of any coil, corresponding to the particle displacement, is along the line of travel of the waves. These are called *Longitudinal Waves*. Pressure waves in a gas (sound waves as they are more generally called) are of this nature, the particle displacement being along the direction of travel of the waves.

It is self-evident that the passage of a transverse wave can be represented by a graph such as Fig. 10.1, for the string of Fig. 10.2 actually takes up the shape of the graph, but it needs more thought before longitudinal waves can be depicted by this method.

10.3 Graphical Representation of Longitudinal Waves

The particle displacement in a longitudinal wave can be represented graphically by the artifice shown in Fig. 10.4. The upper series of strokes represent either the coils of the spring in Fig. 10.3 or layers of air molecules when in the undisturbed position, the lower series shows the displaced position of these layers at some instant when a longitudinal wave motion is travelling through them. The displacement of each layer can be measured, and then plotted at the undisturbed position of the layer in a direction at right angles to the actual displacement.

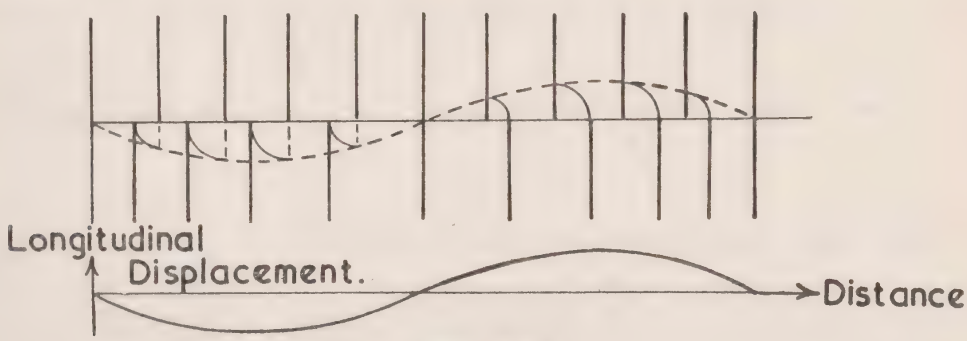


Fig. 10.4

A smooth curve drawn through the points representing the displacement of each layer then gives a graph of longitudinal displacement against distance, but it should be clearly realised that this is not a picture of the wave motion, as are Figs. 10.1 and 10.2, but a graphical representation of the particle displacement which occurs in the wave motion shown pictorially in Fig. 10.3. Notice that a displacement to the left is plotted as negative while a right-hand displacement is positive.

10.4 Pressure Distribution in a Longitudinal Wave Motion in a Gas

So far, a sound wave has been described as a pressure wave travelling in a gaseous medium, but in the previous section a sound wave is treated as a movement of the layers of molecules in the air. Either of these ideas, however, leads to the other. It is clear from Fig. 10.4 that a longitudinal wave motion results in some layers of particles being brought closer together and some moved farther apart, this will cause layers of compression and rarefaction in the medium. The position of these layers can be found as follows.

If a graph of particle displacement is drawn and the direction in which each particle is moving is marked on it as in Fig. 10.5, then it will be seen that at points such as A and C the particles are crowding in towards each other and so building up layers of pressure, while at B the particles are moving outwards and so leaving a region of rarefaction.

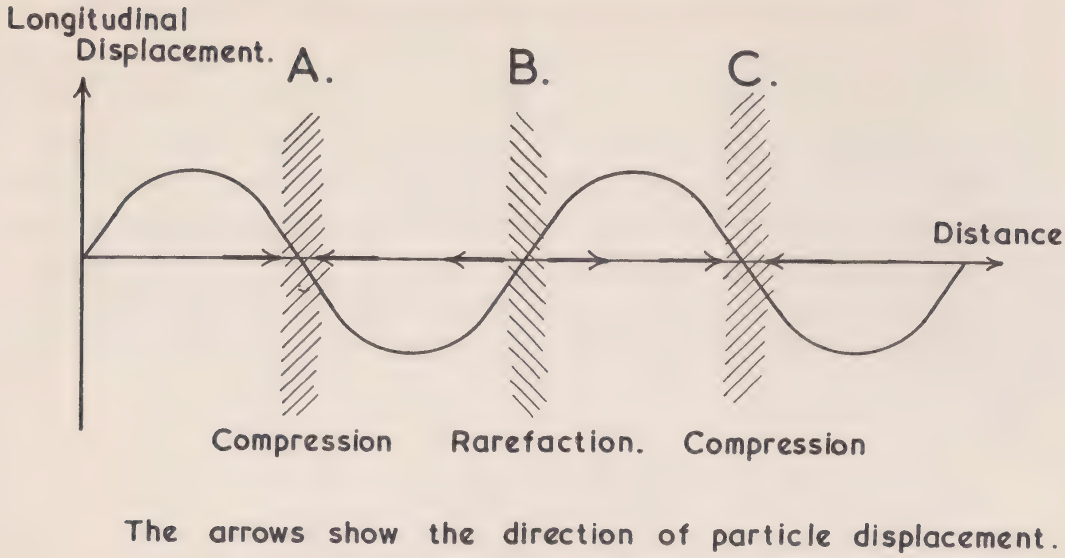


Fig. 10.5

A more exact expression for the distribution of pressure is given below.

Suppose that the particle displacement when a longitudinal wave motion passes through a medium is shown at any instant by the graph in Fig. 10.6.

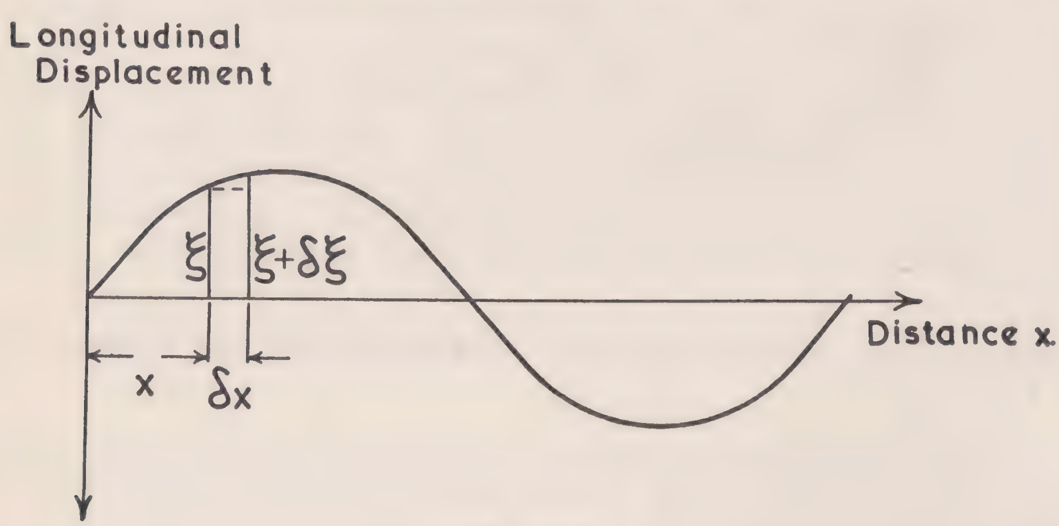


Fig. 10.6

A layer in the medium at a position x receives a longitudinal displacement ξ while one at position $x + \delta x$ is displaced by $\xi + \delta\xi$. Thus a layer of the medium of thickness δx becomes of thickness $\delta x + \delta\xi$ when in the displaced position. If we consider a section of the advancing pressure wave of area of cross-section A, then the volume of gas within the element increases from $A \,.\, \delta x$ to $A(\delta x + \delta\xi)$; thus if the pressure in the undisturbed gas is p_0 the pressure p in the displacement element is given by:

$$p_0\,(A\,\delta x) = p[A(\delta x + \delta\xi)]$$

(from Boyle's Law, $P_1V_1 = P_2V_2$), thus:

$$p_0\,\delta x = p\,(\delta x + \delta\xi)$$

$$\text{or } \delta x(p_0 - p) = p\delta\xi,$$

$$\text{thus } \frac{\delta\xi}{\delta x} = \frac{p_0 - p}{p}.$$

If the *change* in pressure Δp at any point is defined as the difference $p - p_0$,

$$\text{then } \frac{\delta\xi}{\delta x} = \frac{-\Delta p}{p} \quad . \quad . \quad . \quad . \quad (11)$$

Now when a sound wave passes through a gas, the pressure changes are very small indeed—of the order of 1 dyne.cm^{-2} compared with an atmospheric pressure of 10^6 dyne.cm^{-2}. Thus Δp is small compared with p_0, or p and p_0 are very nearly the same, hence one can be replaced by the other in Equation (11) without any sensible error. Making this substitution, Equation (11) leads to:

$$\Delta p = -p_0 \frac{\delta\xi}{\delta x}$$

or, in the limit, as $\delta x \to 0$,

$$\Delta p = -\frac{p_0 d\xi}{dx} \quad . \quad . \quad . \quad . \quad (12)$$

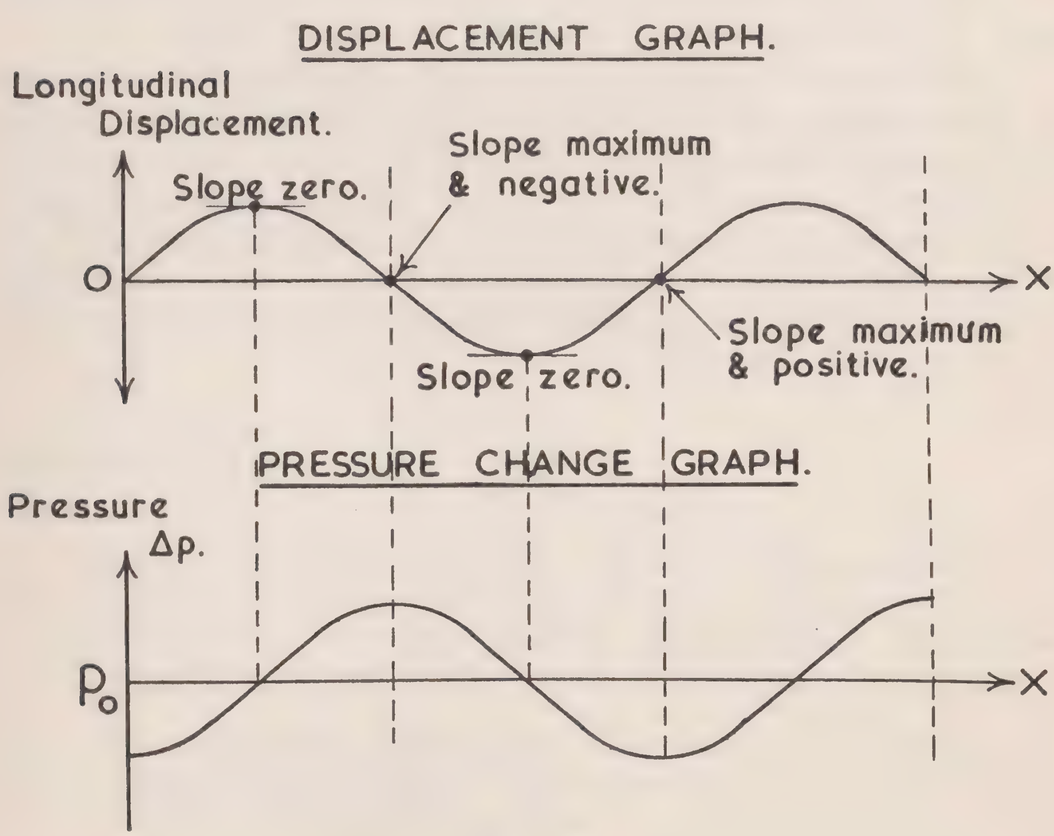


Fig. 10.7

It will be recognised that $\frac{d\xi}{dx}$ is the slope of the particle-displacement graph and thus the pressure changes at any point can be derived from the slope at that point. This can be done graphically as in Fig. 10.7 or as follows.

If the displacement curve is given by

$$\xi = a \sin 2\pi\left(\frac{t}{\tau} - \frac{x}{\lambda}\right),$$

$$\text{then } \frac{d\xi}{dx} = -\frac{2\pi a}{\lambda} \cos 2\pi\left(\frac{t}{\tau} - \frac{x}{\lambda}\right).$$

Thus, from Equation (12), Δp is given by:

$$\Delta p = \frac{2\pi a p_0}{\lambda} \cos 2\pi\left(\frac{t}{\tau} - \frac{x}{\lambda}\right),$$

and the total pressure distribution is given by:

$$p_0 + \Delta p = p_0\left[1 + \frac{2\pi a}{\lambda} \cos 2\pi\left(\frac{t}{\tau} - \frac{x}{\lambda}\right)\right].$$

In general, then, it will be seen that the points of maximum pressure change do not coincide with the points of maximum displacement and in the special case when the displacement curve is a sine wave, then the pressure curve is displaced by a quarter of a wavelength.

It must be emphasised that these results are applicable only when the displacements are small compared with the wavelength and the pressure changes are small compared with the undisturbed pressure in the medium. These conditions are met in the case of sound waves in air, for the value of atmospheric pressure is about 10^6 dyne.cm^{-2} and the wavelength of sound waves is about 1 metre while the pressure changes and particle displacement occurring in a sound wave are as given in the table below.

	Pressure Variation	*Particle Displacement*
Loudest sound that ear can tolerate	300 dyne.cm^{-2}	0·004 cm
Faintest sound that ear can hear	0·0002 dyne.cm^{-2}	3×10^{-9} cm

It is interesting to note that the particle displacement for quiet sounds is smaller than the wavelength of light (5×10^{-5} cm), and for the very faintest of sounds is even smaller than the diameter of an atom (10^{-8} cm).

10.5 Production and Propagation of Sound

Sound has already been introduced as a wave motion in a gas, considered either as a longitudinal particle motion or as a progressive pressure wave.

A sound is generated by anything which imparts such a wave motion to the medium surrounding it. Familiar examples are the tines of a tuning fork or the diaphragm of a loudspeaker, both of which impart a vibrating motion to the air molecules. An explosion involves a sudden generation of gas at high pressure, this expands as a pressure wave, and produces the 'bang' heard by an observer. Again, there are the vocal chords, which are elastic membranes closing the larynx to the passage of air. When air is forced between them they vibrate and break the steady stream of air from the lungs into a series of very rapid pressure pulses which excite sound waves in the surrounding air.

Sound can travel through any medium capable of sustaining a vibratory motion, and in fact passes through solids and liquids as well as gases—we all have experience of sounds passing through solid walls and running along pipes.

The waves in any medium are either particle displacement or pressure waves—it is only the sensation in the brain which occurs when such a wave motion impinges on the ear that can really be described as a sound, nevertheless, the term 'sound-wave' is commonly applied to the wave motion itself.

10.6 The Ear and Hearing

In order to see how a pressure wave is turned into a sound, it is necessary to examine the mechanism of the ear.

Fig. 10.8 shows a section of an ear in diagrammatic form. Pressure

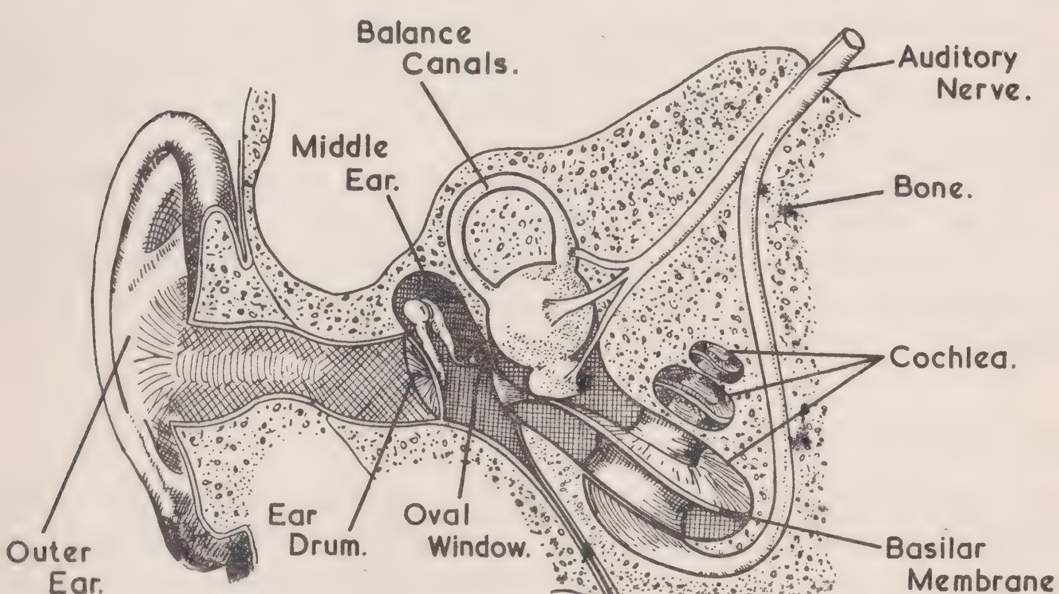


Fig. 10.8

variations in the air cause the ear drum to vibrate, and these vibrations are passed on by three small bones in the middle ear to another membrane called the oval window. These bones, the auditory ossicles, are called the hammer, anvil and stirrup, and are so arranged that they have a large mechanical advantage, and the force applied to the oval window is considerably greater than that acting upon the ear drum.

The oval window leads into the inner ear. This is a cavity within the bony structure of the ear and is shaped like a snail's shell; it is called the cochlea, and is filled with a fluid. Stretched between the walls of the cochlea and coiling up through it almost to its tip is the basilar membrane. This divides the cochlea into upper and lower portions, the oval window being in communication with the upper portion. Thus a movement of the oval window forces the fluid over the surface of the basilar membrane up to the top of the cochlea, around the upper end of the membrane (for it is not joined to the cochlea at its tip), and then down the lower portion of the cochlea. Here the fluid motion is absorbed by the round window, which is another membrane and bulges to allow this fluid motion to take place.

The basilar membrane consists of a vast number of fibres stretched across its width, the fibres vary in length, being long at one end of the membrane and short at the other. Each fibre appears to pick up a vibration from the liquid in the cochlea over some narrow band of frequencies. Several fibres at a time are connected to each nerve ending of the auditory nerve. We believe that a pressure wave at a certain frequency entering the ear causes the ear drum, and hence the fluid in the cochlea, to vibrate at the same frequency. This excites some particular range of the fibres in the basilar membrane into vibration and thus a message is sent via the auditory nerve to the brain, which interprets the sensation as a sound of a particular pitch (see Section **10.12**). If the pressure wave reaching the ear becomes more intense, the fluid motion in the cochlea and hence the motion of the fibres in the basilar membrane becomes violent, this produces a larger nerve impulse and the brain registers a louder sound.

This theory of the operation of the ear leaves many effects still unexplained, but it has at least been established that if a portion of the fibres in the middle of the basilar membrane is removed, then the subject becomes deaf to a band of frequencies in the middle of the aural range.

10.7 Velocity of Waves

Previously it has merely been assumed that a wave motion has a velocity; the velocity is very much a property of the medium in which the waves are travelling and varies with the nature of the waves. Each case has to be examined individually and usually the mathematical

treatment necessary is beyond the scope of this book; in the majority of cases we must content ourselves with a mere statement of the results.

(a) Transverse Waves in a String

Consider a wave of small amplitude travelling with velocity v along a stretched cord; if at the same time the cord is moved in the opposite direction with an equal velocity, the wave will stand still in space, and an equation for the velocity v can be obtained by applying familiar ideas of dynamics to the particles of the cord. A small element δs of

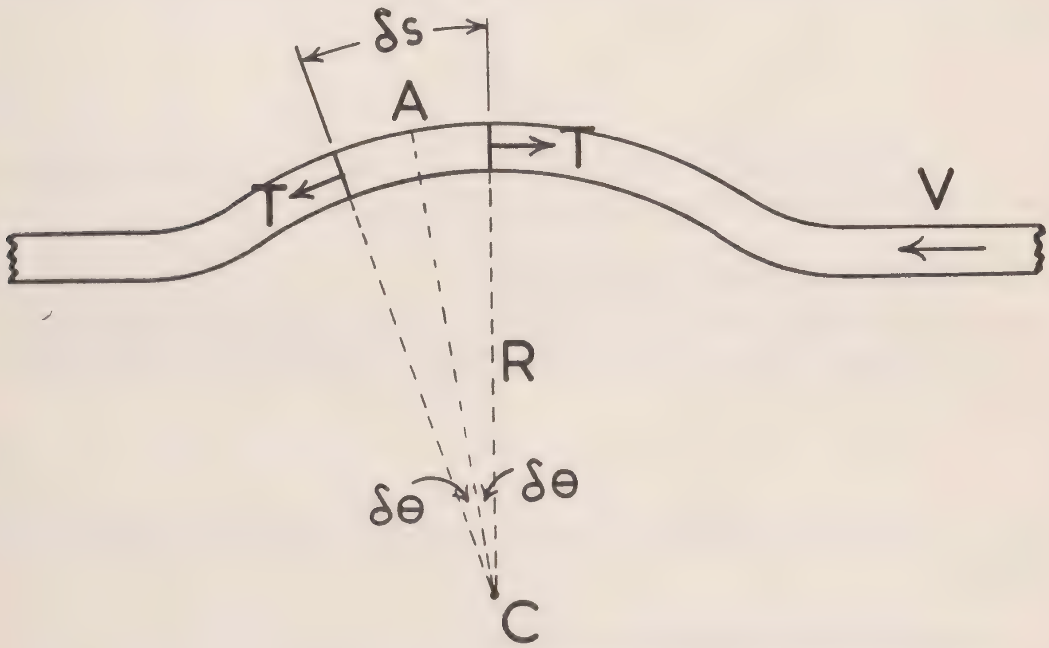


Fig. 10.9

the cord is shown in Fig. 10.9; if this element is very short, it can be considered as an arc of a circle of radius R and subtends an angle $2\delta\theta$ at the centre of curvature C. Provided that the cord is *completely flexible*, the only forces exerted by the rest of the cord on the element will be the tension T acting at each end of the element, and these forces are both inclined at an angle $\delta\theta$ to the central radius AC. Each force thus has a component $T \cos \delta\theta$ perpendicular to AC and a component $T \sin \delta\theta$ acting along AC. The perpendicular components will cancel each other out, but the others will reinforce; thus there is a force $2T \sin \delta\theta$ acting on the element in a direction AC due to the tension in the cord.

The element is, however, instantaneously moving around a circle of radius R and thus must be subject to a centripetal force mv^2/R, where m is the mass of the element; this force is provided by the inward component of the tension, thus

$$2T \sin \delta\theta = \frac{mv^2}{R}$$

$$\text{or } v = \sqrt{\frac{2TR \sin \delta\theta}{m}}.$$

But if $\delta\theta$ is made vanishingly small, we may write $\delta\theta$ for sin $\delta\theta$, also $\delta s/R = 2\delta\theta$. Making these substitutions in the equation above leads to:

$$v = \sqrt{\frac{T\delta s}{m}}.$$

Further, if the mass per unit length of cord is μ, then $m = \mu\delta s$, thus for transverse waves in a stretched string or wire, the velocity is given by

$$v = \sqrt{\frac{T}{\mu}} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where T is the tension in the cord and μ is the mass per unit length of the cord, sometimes called the linear density.

(b) Longitudinal Waves in a Rod

For longitudinal waves moving in one direction in a solid, as, for example, sound waves transmitted along a solid rod, the velocity is given by

$$v = \sqrt{\frac{Y}{\varrho}} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where Y is Young's Modulus for the material of the rod and ϱ is its density.

(c) Longitudinal Waves in a Fluid

If waves move outwards in all directions from a source situated in a fluid, the velocity of the wave motion is given by:

$$v = \sqrt{\frac{K}{\varrho}} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where K is the bulk modulus of the fluid.

(d) Longitudinal Waves in a Gas

In the especial case where the fluid is a gas, instead of Equation (15), the following should be used:

$$v = \sqrt{\frac{\gamma p}{\varrho}} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

for it was shown on page 223 that the bulk modulus of a gas is equal to its pressure, and that the adiabatic bulk modulus is equal to γp; the latter value is used here, for in a sound wave the pressure fluctuations take place much faster than heat can enter or leave the gas; in fact, the sound wave probably represents the nearest approach to adiabatic conditions that can be realised experimentally.

10.8 Velocity of Sound in Air

If m is the mass of a volume V of the gas in which sound waves are travelling, then the density of the gas is given by $\varrho = m/V$; substituting this value in Equation (16) gives:

$$v = \sqrt{\frac{\gamma}{m} \cdot pV}\ .$$

For changes of pressure at constant temperature, pV remains constant, thus the velocity of sound in a gas is independent of the gas pressure; also

$$pV = RT$$

and substituting this in the equation above leads to:

$$v = \sqrt{\frac{\gamma R}{m} \cdot T} \quad . \quad . \quad . \quad . \quad (17)$$

The factor $\gamma R/m$ is a constant, therefore the velocity of sound in a gas is proportional to the square root of the absolute temperature. This may be written as

$$\frac{v_t}{v_0} = \sqrt{\frac{273 + t}{273}},$$

where v_t, v_0 are the velocities at $t°$ C and $0°$ C respectively.

$$\text{Thus } \frac{v_t}{v_0} = \left(1 + \frac{t}{273}\right)^{\frac{1}{2}}.$$

If $t \ll 273$, this may be expanded by the binomial theorem, giving:

$$\frac{v_t}{v_0} = \left(1 + \frac{t}{546}\right)$$

$$v_t = v_0\left(1 + \frac{1}{546}t\right) \quad . \quad . \quad . \quad (18)$$

Using this equation, experimental results collected under varying conditions of temperature can be reduced to a standard temperature.

If the values of γ, p and ϱ for air are inserted in Equation (16), then the velocity of sound in air is found to be about 332 m.sec^{-1} at 0° C and increases by a further 0·6 m.sec^{-1} for every Centigrade degree rise in temperature. At room temperature the velocity of sound is about 1100 ft.sec^{-1}.

The frequencies occurring in ordinary sounds cover a large range—from 10 cycles per second up to 20,000 cycles per second. Inserting these values and the value for the velocity of sound in Equation (6) indicates that the wavelength of sound waves range from 33 metres down to 1·5 cm. 1000 cps is often used as a standard frequency, giving a wavelength of about 34 cm at room temperatures (the abbreviation cps is used for 'cycles per second').

Attempts to measure the velocity of sound directly were made as long ago as the sixteenth century and have been repeated many times since. All of these experiments turned on measuring the time interval between seeing the flash of a distant cannon firing and hearing the report. If the distance to the cannon is known, the velocity of sound can be calculated. The method suffered from a number of disadvantages, of which the most important were firstly the time-lags of the observers operating the timing devices, and secondly the presence of a wind, which modifies the speed of sound (see Section 10.9).

The first effect is removed nowadays by recording electrically the firing of the gun and the subsequent arrival of the report. The disturbing influence of wind can be removed if two readings are taken in opposite directions over the same range; the wind then assists the passage of the sound wave in one direction but opposes it in the other, so that the average of the two readings gives the true velocity of sound. The two readings must, of course, be taken simultaneously or the speed of the wind may change between the two experiments.

Reynault in 1862 tried to avoid the errors due to wind by experimenting in the large pipes which were then being laid as the water mains of Paris.

Despite all of these refinements, the method is neither convenient nor capable of high accuracy because of the uncertainty of conditions obtaining in the air over the long distances used, and nowadays is superseded by methods to be described in the following chapters.

Example 1. *Find the ratio of the velocity of sound in air to the velocity of sound in water at* 15° *C, using the following data:*

Density of air at S.T.P. $= 1{\cdot}293$ gm per litre.
Ratio of specific heats for air $= 1{\cdot}41$.
Density of mercury $= 13{\cdot}6$ gm.cm^{-3}.
Bulk modulus of water $= 2{\cdot}23 \times 10^{10}$ dyne.cm^{-2}.
Density of water at 15° C $= 1{\cdot}00$ gm.cm^{-3}.

The velocity of sound in air at S.T.P. is given by

$$v_0 = \sqrt{\frac{\gamma p}{\varrho}}$$

where the symbols have the usual significance,

$$\text{hence } v = \sqrt{\frac{1{\cdot}41 \times 76 \times 981 \times 13{\cdot}6}{1{\cdot}293}} \text{ cm.sec}^{-1}.$$

(Note the conversion of pressure from 'cm of Hg' to dyne.cm^{-2}.)

$$\begin{aligned} \text{Thus } v_0 &= 332 \text{ m.sec}^{-1}. \\ \text{Now } v &= v_0\,(1 + t/546) \\ \text{hence } v_{15} &= 332\,(1 + 15/546) \\ &= 341 \text{ m.sec}^{-1}. \end{aligned}$$

The velocity of sound in water is given by

$$
\begin{aligned}
v &= \sqrt{\frac{K}{\varrho}} \\
&= \sqrt{\frac{2{\cdot}23 \times 10^{10}}{1}} \text{ (at } 15^\circ \text{ C)} \\
&= 1490 \text{ m.sec}^{-1}.
\end{aligned}
$$

Thus the ratio of the velocities is 341 : 1490 or 1 : 4·38.

10.9 Doppler Effect

The results derived in the preceding paragraph make no allowance for a bodily motion of the medium in which the waves are travelling, or for any motion of the source of the sound, both of which modify the nature of the wave motion. Also, if the observer is in motion, the sensation that he receives differs from that when he is at rest. Doppler, amongst others, noted this effect early in the nineteenth century—not relating to sound waves, as it happens, but when seeking an explanation of a similar effect observed with light waves.

(a) Moving Source

A source at rest emits a sound wave of frequency f, and of wave-length λ. If this wave motion travels with a velocity v, then from Equation (6),

$$v = f\lambda.$$

If, however, the source is moving, the effect on the wave motion can be calculated by noticing what happens to the waves emitted in 1 second.

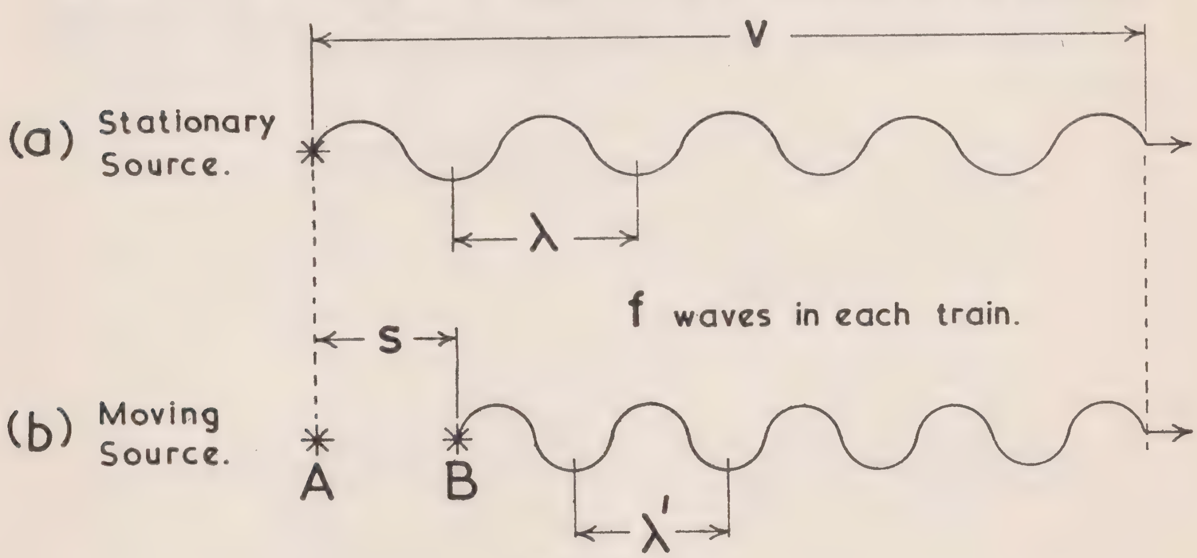


Fig. 10.10

The source will emit f waves in one second when either stationary or moving; in both cases these waves travel with a velocity v. If the source is stationary, the first wave emitted will be at a distance v from the source at the end of 1 second and the subsequent waves up to the fth

will occupy the distance v back to the source, but if the source itself moves from A to B in 1 second (Fig. 10.10), the first wave emitted while the source is at A will still travel a distance v from A, but the fth wave will only just be emitted as the source reaches B. If the velocity of the source is s, then AB is equal to s, and f waves will therefore be packed into a distance $(v - s)$; they will thus have a new wavelength λ' given by

$$\lambda' = \frac{(v - s)}{f} \quad . \quad . \quad . \quad . \quad (19)$$

This wavetrain is still moving forward with a velocity v, and thus, as it sweeps past an observer, he will note the passage of one wavelength in a time λ'/v and he will interpret the wave as having a frequency f' given by

$$f' = \frac{v}{\lambda'}$$

or, substituting for λ' from Equation (19):

$$f' = f\left(\frac{v}{v - s}\right) \quad . \quad . \quad . \quad . \quad (20)$$

The frequency of the sound heard by the observer is therefore higher than the time frequency of source by the ratio

$$\left(\frac{\text{Velocity of sound}}{\text{Velocity of sound} - \text{Velocity of source}}\right)$$

The motion of the source has to be in the same direction as the wave motion for this equation to be applicable. It is, however, generally true if motion of the source away from the observer is ascribed a negative value.

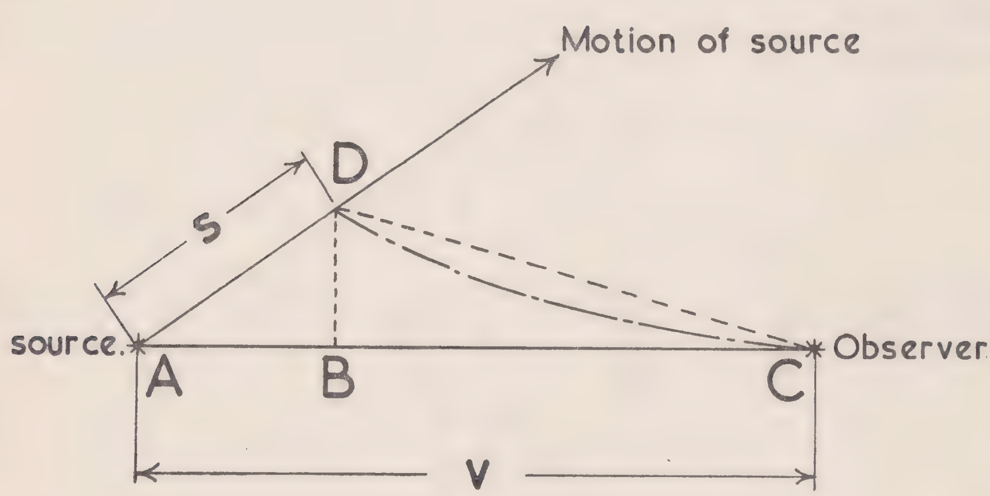


Fig. 10.11

If the motion of the source is not along the line joining source and observer, but is along a line making an angle θ with this direction, then

the expression for change of frequency is rather more complicated. It can be calculated as follows.

Let the source emit sound waves of frequency f which travel with a velocity v, while the source travels with a velocity s along the line AD (Fig. 10.11). The train of sound waves which it emits in one second will lie along a curved path such as that shown between D and C. Now if the velocity of the source is much smaller than the velocity of sound, then BD will be much smaller than BC, and the length of the curved path will be very nearly equal to BC.

$$\text{But } BC = v - s\cos\theta,$$

thus the length of the wavetrain $\simeq v - s\cos\theta$,

and following the same argument as that used above leads to:

$$\text{new wavelength } \lambda' = \frac{v - s\cos\theta}{f}.$$

Hence the new frequency f' heard by a stationary observer is given by:

$$f' = f\left(\frac{v}{v - s\cos\theta}\right) \qquad . \qquad . \quad (21)$$

Comparing this with Equation (20) we see that s is replaced by $s\cos\theta$ where $s\cos\theta$ is the component in the direction of the observer of the velocity of the source. When $\theta = 90°$, i.e. the source is moving at right angles across the line to the observer, then $\cos\theta = 0$ and $f' = f$ so that no change in frequency is heard. A familiar example of this effect is the fall in frequency of a train whistle often heard as an express passes through a station. Suppose that the true frequency of the whistle is 1000 cps and the train is travelling at 100 ft.sec^{-1} (about 70 mph), then, when the train is approaching from a distance, the frequency heard is given by:

$$f' = 1000\left(\frac{1100}{1100 - 100}\right)$$

(since the velocity of sound in air is 1100 ft.sec^{-1}).

$$\text{or } f' = 1100 \text{ cps}.$$

As the engine passes the observer, its motion is at right angles to the line joining source and observer, and so the frequency falls to 1000 cps; but as it recedes into the distance the frequency is now given by

$$f' = 1000\left(\frac{1100}{1100 + 100}\right)$$

$$\text{or } f' = 917 \text{ cps}.$$

Thus, whilst approaching, the note of the whistle is heard rather higher than its true frequency, but this falls to a much lower value as the train sweeps by. The rate at which the note falls depends on how close the observer stands to the track. This is illustrated in Fig. 10.12.

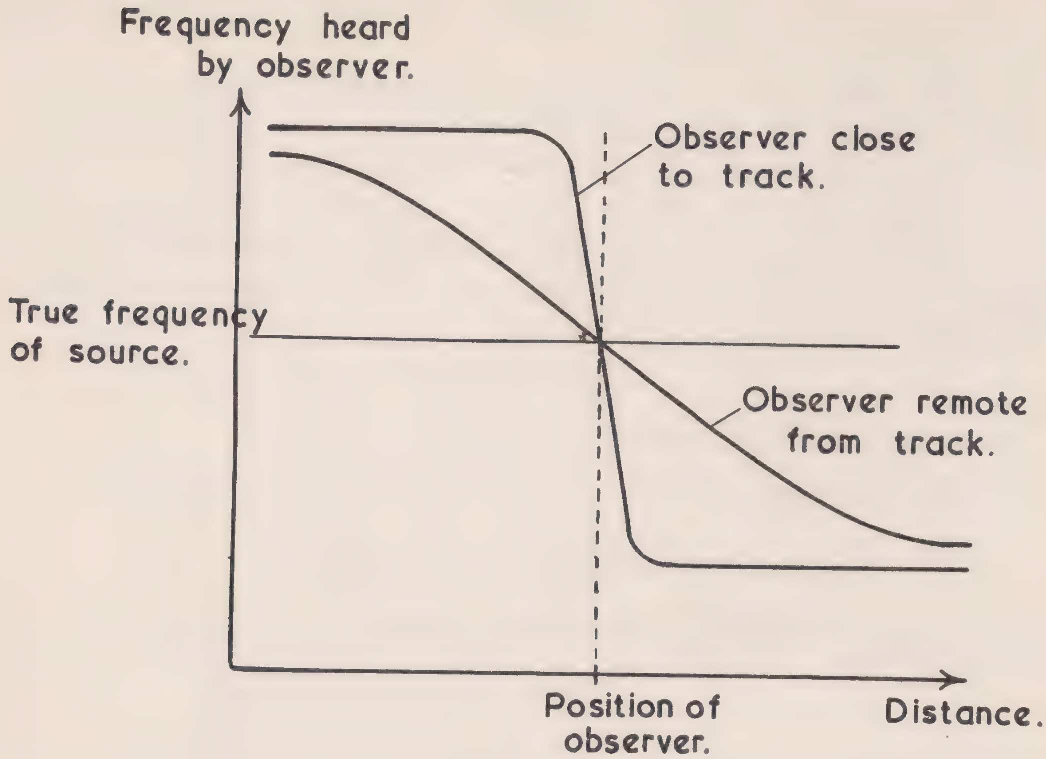


Fig. 10.12

(b) Moving Observer

When the observer is in motion, the effect of his motion can be calculated, as before, by considering the number of waves emitted by the source and passing the observer in one second.

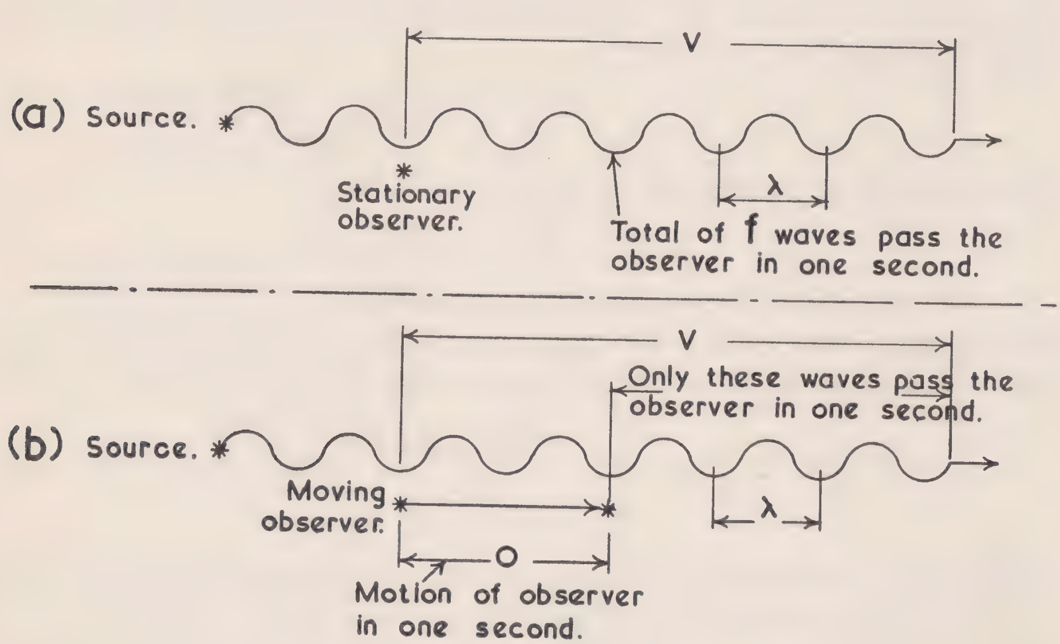


Fig. 10.13

If the source emits a note of frequency f and of wavelength λ, then the wavetrain will travel with a velocity v given by

$$v = f\lambda.$$

In one second, f waves will pass an observer if he is stationary and these f waves will occupy a distance v (Fig. 10.13 (*a*)).

If, however, the observer is moving in the same direction as the wavetrain with a velocity o (Fig. 10.13 (*b*)), then in one second he will move a distance o, and the waves which have overtaken him will occupy a distance $(v - o)$. Since the wavelength is unchanged by the motion of the observer, the number of waves which pass the observer in one second will be $(v - o)/\lambda$. This is the frequency of the note that the observer hears, or

$$f' = \frac{v - o}{\lambda}.$$

Substituting for λ from the equation above gives

$$f' = f\left(\frac{v - o}{v}\right) \quad . \quad . \quad . \quad . \quad (22)$$

It should be noted that the velocity o is accounted positive if it is in the same direction as the wavetrain, and negative if in the opposite direction. If the observer moves at an angle to the line between source and observer, then an expression similar to Equation (21) could be developed.

The motion of the observer does not have the same effect on the frequency of the note that he hears as the motion of the source. It has been seen above that if a source of 1000 cps approaches an observer at 100 ft.sec^{-1}, then the note heard is 1100 cps. If, however, the observer approaches the source at 100 ft.sec^{-1}, substituting these values in Equation (22) gives the frequency of the note heard as:

$$\begin{aligned} f' &= 1000 \times \left(\frac{1100 + 100}{1100}\right) \\ &= 1091 \text{ cps}, \end{aligned}$$

i.e. 9 cps lower than in the previous case. The effect is even more marked when source and observer move with the velocity of sound, i.e. $s = o = v$ in Equations (20) and (22). The resultant frequency heard by the observer is given in the following table.

True frequency of source = 1000 *cps*	*Approaching*	*Receding*
Source moving	∞ cps	500 cps
Observer moving	2000 cps	0 cps

(c) Moving Medium

If the medium in which the sound waves are moving is itself in motion (i.e. a wind is blowing) then both the wavelength and the speed at which a wavetrain passes a stationary observer will be altered.

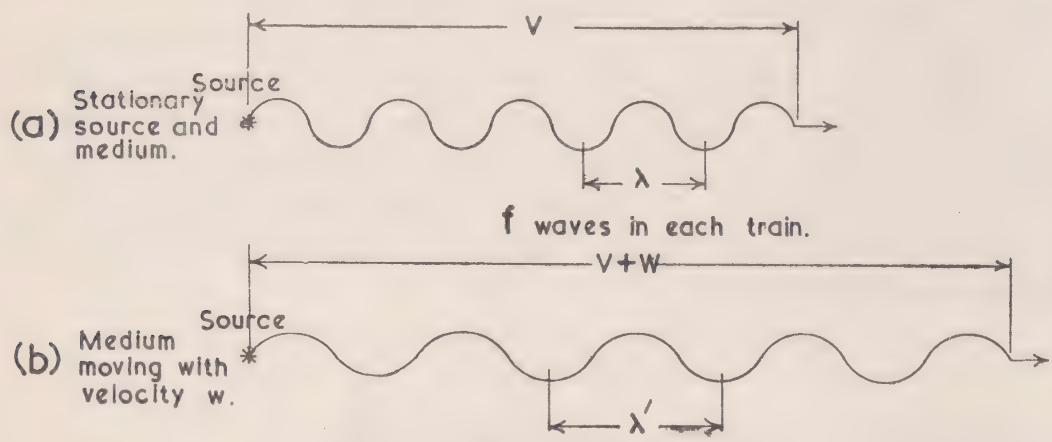


Fig. 10.14

For example, if the medium is moving with a velocity w in the direction of the wavetrain, then in one second a wave emitted by the source will travel a distance v through the medium. At the same time the medium will move on a distance w, consequently the wave will be at a distance $(v + w)$ from the source in one second (Fig. 10.14 (b)).

The wavetrain thus travels at a velocity $(v + w)$ and if the frequency of the source is f, has a wavelength given by

$$\lambda' = \frac{v + w}{f} \quad . \quad . \quad . \quad . \quad (23)$$

The frequency of the note heard by a stationary observer is not changed, however, for both velocity and wavelength are changed in the same ratio $(v + w)/w$. The observed frequency is given by the velocity of the wavetrain divided by its wavelength, thus:

$$\text{Observed frequency} = \frac{v + w}{\lambda'}$$

and this is equal to the true frequency of source from Equation (23).

(d) Moving Source or Observer in a Moving Medium

The presence of a wind modifies the velocity and wavelength of a wavetrain, but the same effects can be produced if the wavetrain is assumed to move through a still medium at the increased velocity $(v + w)$. Thus if either a source or an observer is moving in a moving medium, the change in observed frequency can be found by substituting $(v + w)$ for v in either of the equations developed for still air, i.e. in Equations (20) or (22).

Thus for a moving source we have

$$f' = f\left(\frac{v + w}{v + w - s}\right) \quad . \quad . \quad . \quad (24)$$

and for a moving observer

$$f' = f\left(\frac{v + w - o}{v + w}\right) \quad . \quad . \quad . \quad (25)$$

If all three happen at once, i.e. moving source, moving observer and a wind blowing, then, by applying the methods used above,

$$f' = f\left(\frac{v + w - o}{v + w - s}\right) \quad . \quad . \quad . \quad (26)$$

Although the description of the Doppler effect given above is restricted to sound waves, it occurs with any form of wave motion. For example, the wavelength of the light emitted by a source is influenced by the motion of the source. Owing to the high velocity of light (approx. 300,000 kilometres.sec^{-1}) the effect becomes apparent only when the source itself is moving at a comparable velocity.

The only objects capable of such high speeds are gaseous molecules and their constituents and celestial bodies; thus light emitted from a star moving away from the solar system would appear to an observer on the Earth to be of longer wavelength (i.e. displaced towards the red end of the spectrum) than light from the same star at rest. It is by measuring this increase in wavelength that the velocities of some of the stars have been calculated.

Radio waves are also a wave motion and the Doppler effect becomes apparent when radio waves are reflected from a moving object such as an aeroplane in flight. This phenomenon is used in some radar devices to measure the velocity of the aeroplane.

Example 2. *A car has a pair of horns differing in frequency by* 20 *cps. Calculate the difference in frequency of the two notes heard by a stationary observer when the car approaches him at* 45 *mph. (The velocity of sound in air is* 1100 *ft.sec*$^{-1}$).

Let the lower frequency of the two horns be f and the higher $f + 20$ cps. Then the lower note heard by the observer is $f\left(\frac{v}{v - s}\right)$ and the higher note $(f + 20)\left(\frac{v}{v - s}\right)$, where v is the velocity of sound and s the velocity of the car.

Thus the *difference* in the two frequencies heard by the observer is given by:

$$\text{difference} = (f + 20)\left(\frac{v}{v - s}\right) - f\left(\frac{v}{v - s}\right)$$

$$= 20\left(\frac{v}{v - s}\right).$$

Hence, in this case:

$$\begin{aligned}\text{difference} &= 20\left(\frac{1100}{1100 - 66}\right)\\ &= 20 \times \frac{1100}{1034}\\ &= 21{\cdot}3 \text{ cps.}\end{aligned}$$

10.10 Beats

The earliest musical experience received by most of us is the discordant and jarring sound that occurs when a rather inexpert finger hits the crack between two keys on the piano and two notes are sounded together—the phenomenon is rather aptly described as two sounds *beating* together. If the two notes are very nearly of the same frequency, i.e. differing by a few cycles, then the discordant nature of the resultant sound is much reduced; instead it now pulsates in loudness. This pulsation is accounted for in the following paragraph, but the discordance is treated again in Chapter 12.

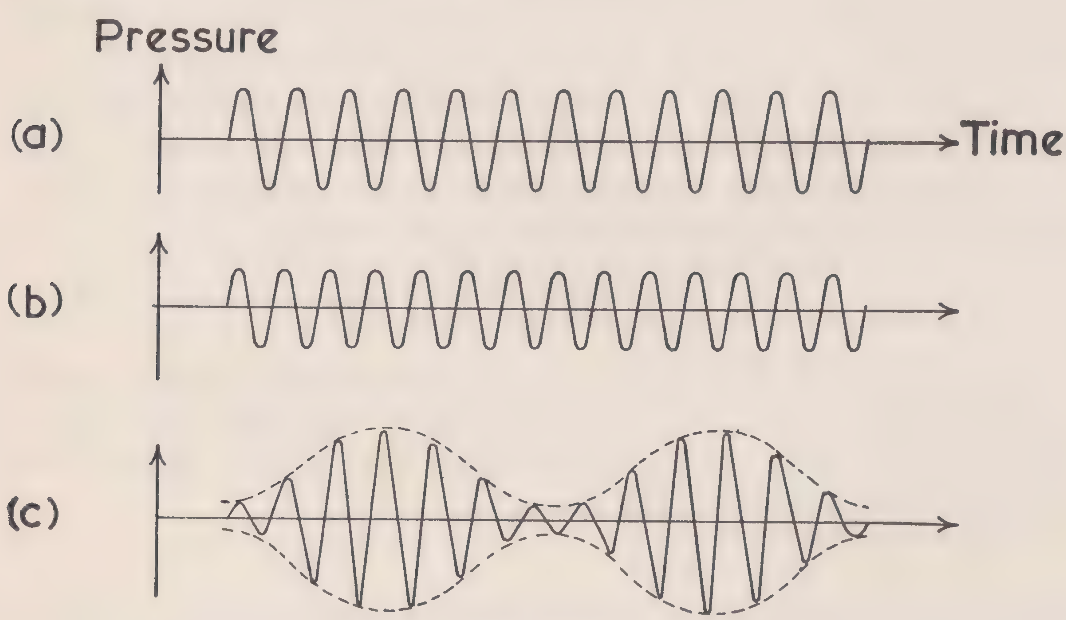


Fig. 10.15

If two sources of sound emit pressure waves as shown in Fig. 10.15 (*a*) and (*b*), then the resultant pressure wave received by the ear is the sum of these two wavetrains. This can be found by adding together the ordinates of each curve at the same instant and plotting therefrom the resultant curve shown in Fig. 10.15 (*c*). It will be seen that this new curve is of very nearly the same frequency as either of the original curves (the exact frequency is derived later, see page 304) but the amplitude varies as the two waves go in and out of step.

A loud sound is heard when two pressure pulses arrive at the ear at the same instant and so reinforce each other; but if the pressure

pulse due to one wavetrain coincides with a rarefaction due to the other they will cancel each other out and the sound will momentarily become very quiet. If the two original waves are of the same amplitude, they cancel each other out entirely at certain instants, and a moment of complete silence occurs between each loudening of the sound. In general, however, the sounds are of different amplitude and absolute cancellation never occurs.

The frequency with which the loudness pulsates is equal to the difference in frequency of the two sources, for if one emits n more waves per second than the other, it will catch up one whole cycle in each $1/n$ sec, hence the two wavetrains will reinforce each other n times in a second.

The addition of two pressure waves can be seen mathematically as follows.

Let the pressure in one waveform at any point be given by:

$$p_1 = p_0 \sin 2\pi f_1 t$$

and in the other

$$p_2 = p_0 \sin 2\pi f_2 t,$$

where both the waves have the same amplitude p_0, but different frequencies f_1 and f_2.

The total pressure is the sum of these two, i.e.

$$\begin{aligned} p &= p_1 + p_2 \\ &= p_0 (\sin 2\pi f_1 t + \sin 2\pi f_2 t) \\ &= p_0 \left[2 \sin\left\{2\pi\left(\frac{f_1 + f_2}{2}\right)t\right\} . \cos\left\{2\pi\left(\frac{f_1 - f_2}{2}\right)t\right\}\right] \\ &= 2p_0 \cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t \,.\, \sin 2\pi\left(\frac{f_1 + f_2}{2}\right)t. \qquad (27) \end{aligned}$$

This represents a sinusoidal variation in pressure, due to the factor $\sin 2\pi\left(\frac{f_1 + f_2}{2}\right)t$, at a frequency of $\frac{f_1 + f_2}{2}$, i.e. the average of the two original frequencies. The amplitude is not constant but is given by the term $2p_0 \cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t$; this grows and decays with a frequency $\frac{f_1 - f_2}{2}$. The amplitude, however, goes through two maxima in each cycle, hence the 'beat frequency' is $f_1 - f_2$, that is, the difference between the two frequencies.

10.11 Methods of Measuring Frequency

The frequency of a sound is one of the more difficult quantities to be measured in a laboratory, and it is only in the last decade that really reliable methods have been developed for use by acoustical engineers.

Most of the methods used in the laboratory only compare an unknown frequency with that of a standard source, such as a tuning-fork. The two absolute methods described below, the dropping plate and the siren, are unfortunately of limited application and low accuracy. Frequency meters, which overcome these difficulties, are discussed in Chapter 12.

(a) Falling-plate Method

This method is applicable only to the measurement of the frequency of some source which makes fairly violent oscillations, such as the tines of a tuning-fork; it is only with great difficulty that the method can be adapted to measure the frequency of a sound wave in air.

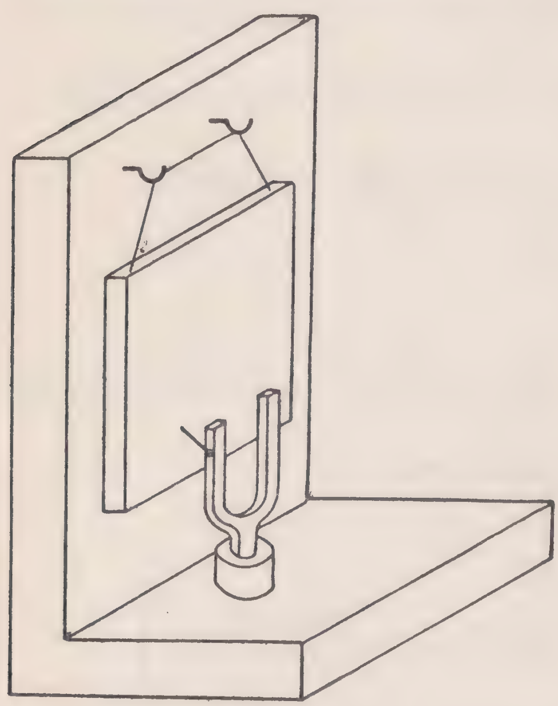


Fig. 10.16

A glass plate is blackened with smoke and suspended by a piece of cotton as shown in Fig. 10.16. Notice that this figure is purely diagrammatic, many varieties of this piece of apparatus are available and the reader is recommended to examine carefully the one available in his own laboratory.

A bristle is fixed to one tine of the fork with wax and adjusted so that it presses lightly against the plate; the fork is then sounded by bowing with a well-rosined bow and the cotton holding the plate is burnt. The plate drops, falling freely under gravity if adjusted properly, and the bristle traces a wavy curve on the smoked surface.

The trace is placed under a vernier microscope, the axis of which is adjusted by eye to coincide with the centre line of the curve.

If now a distance s is measured from the start of the trace, the time taken to fall this distance is given by:

$$s = \tfrac{1}{2}gt^2$$

$$\text{or } t = \sqrt{\frac{2s}{g}} \qquad . \quad . \quad . \quad . \quad (28)$$

If this distance contains N whole vibrations of the fork, then the time for one vibration is t/N and the frequency f is N/t, thus:

$$f = N\sqrt{\frac{g}{2s}} \qquad . \quad . \quad . \quad (29)$$

For refinements of this experiment, the student is referred to *Experimental Physics*, by Daish and Fender, published by English Universities Press, Ltd.

(b) The Siren

The construction of a siren is shown in Figs. 10.17 and 10.18.

If air is forced into the wind chest and the disc set spinning, a small puff of air will escape whenever the sets of holes in the wind chest and disc coincide. When the disc is spinning quickly these puffs will occur so rapidly after each other that a note will be heard.

If the plate does r revolutions in 1 second and is drilled with n holes, then nr puffs of air will be emitted in one second, and this will be the frequency of the note heard or:

$$f = nr \qquad \ldots \qquad (30)$$

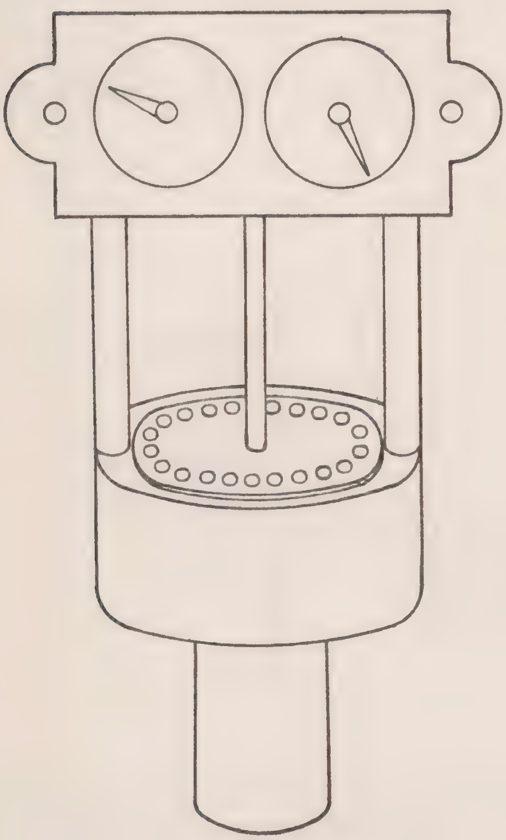


Fig. 10.17

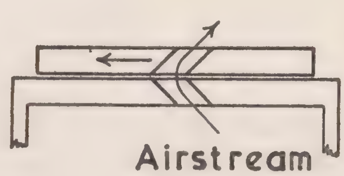


Fig. 10.18

The inclined holes in the spinning disc make it act as a reaction turbine and the plate derives enough impulse from the outgoing air to keep it spinning once started, in fact its speed can be adjusted by varying the air pressure.

To measure the frequency emitted by a source of sound, the siren is operated and its speed adjusted until it emits the same note as the source. The speed at which the plate is spinning is then measured by means of a stopwatch and a revolution counter geared to the shaft; this gives the value of r in Equation (30). The number of holes in the plate can, of course, be counted directly, and substitution of these values in Equation (30) gives the frequency of the source.

It is not easy to adjust the siren to give exactly the same note as the source and beats will generally be heard between the two notes. If this is the case, the frequency of the source is found as follows. Let the disc be timed over t seconds and in this time make R revolutions; then

the total number of cycles emitted by the siren is Rn. If, however, a total of b beats are heard between the two notes in this time, then the source emits $(Rn \pm b)$ cycles (the ambiguity of sign arises since the number of beats per second is only equal to the difference between the two frequencies—either can be the higher). Thus the frequency of the source is given by:

$$f = \frac{Rn \pm b}{t} \text{ cps.} \qquad (31)$$

To find which sign is the appropriate one in an actual case, consider the diagram in Fig. 10.19.

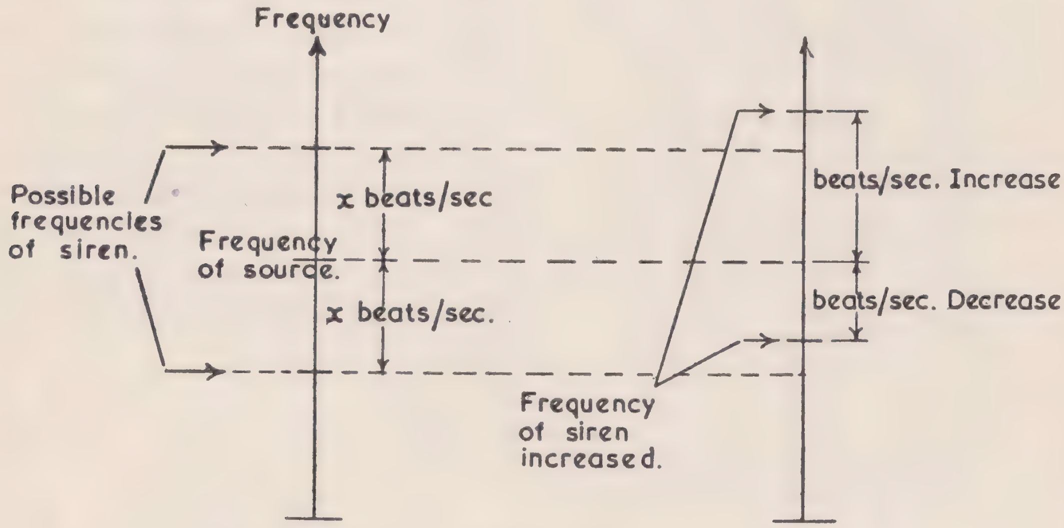


Fig. 10.19

This shows that if the frequency of the siren is increased by a *very small* amount, then the rate of beating will increase if the frequency of the siren is higher than the frequency of the source and decrease if it is lower; by this test the correct sign in Equation (31) can be chosen.

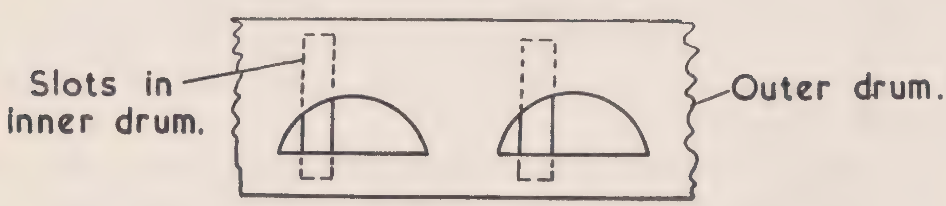


Fig. 10.20

It will be seen that the output of the siren is a series of pulses, not a sine wave; this gives rise to a very complex series of overtones (see Chapter 12). Further, the wind pressure governs both the loudness of the note and speed of the disc, thus these two factors are interrelated. In modern models of the siren, the two discs are replaced by drums fitting one inside the other. The inner drum has slots cut in it, while the outer one has holes shaped as in Fig. 10.20. The inner drum is spun at a

fixed speed by an electric motor which also compresses the air, this issues through the slots and produces a note of nearly sinusoidal waveform.

Example 3. *The disc of a siren is driven by a synchronous electric motor running at 1500 rpm, and the note of the siren makes 20 beats in 9 seconds with the note of a tuning-fork. When the motor is first switched off, the rate of beating is observed to slow down and then increase again. If the disc has 24 holes, what is the frequency of the fork?*

The disc makes 1500 rpm, i.e. 1,500/60 or 25 rev. in one sec, therefore the frequency of the siren note is $25 \times 24 = 600$ cps.

The note of the siren makes $20/9 = 2{\cdot}2$ beats per sec with the note of the fork, hence the frequency of the fork is $600 \pm 2{\cdot}2$ cps.

When the motor is switched off, the speed of the disc decreases and the frequency of the siren falls, but the rate of beating also falls; this means that the two frequencies are coming closer together. Thus the siren must have had the higher frequency and the frequency of the fork is 597·8 cps.

Notice that as the frequency of the siren falls *below* 597·8, the beat frequency will rise again.

10.12 Pitch and Frequency

A sound wave of fixed frequency impinging on our ears gives rise to a sensation in our brain which we describe as a musical note; every frequency produces a specific note, each of which could quite adequately be described by the frequency of the sound wave which produces it. Unfortunately musicians have adopted a different nomenclature to describe this mental sensation: they talk about the *pitch* of the note and describe the pitch by a series of letters. Thus a sound wave of frequency 256 cycles per second gives rise to a musical note whose pitch is said to be 'middle C'. A table relating pitch and frequency is given below but this question is discussed more fully in a later chapter.

Frequency	*Pitch*
256 cps	Middle C
288	D
320	E
342	F
384	G
426	A
480	B
512	Top C

It is sufficient to note that certain frequencies are chosen by musicians, and the pitch of each one is given a specific name (*A*, *B*, *C* to *G*); such a series of notes is called a *musical scale.* The lettering system begins to repeat itself when a frequency double that of the original note is reached; the musical interval between two notes

designated by the same letter is called 'an octave'—consequently a change in pitch of one octave represents a doubling of the frequency of the note.

It will be seen here that pitch and frequency are units of a very different kind—the frequency of a note can be measured just as any other physical quantity, while the pitch of a note only describes the mental sensation experienced by an observer—it cannot be measured without the co-operation of the subject. Pitch is called a *physiological quantity* whereas frequency is a *physical quantity*.

Many other quantities can be described either by physical or physiological units; examples are the wavelength or the colour of light, temperature or warmth, and the intensity or the loudness of a sound.

10.13 Intensity and Loudness

It was explained on page 278 that the energy transmitted by a sound wave could be used as a measure of its *intensity*; this is a physical quantity and can be measured with suitable apparatus of great sensitivity.

The *loudness* of the sound is a physiological unit and cannot be measured directly—it is quite obvious that a sound described as loud by one person may not be described in similar terms by another. The only assessment of loudness that the ear can make reliably is to judge when two sounds are of *equal* loudness.

10.14 Decibels and Phons

The ear, in common with most of our sensory organs, is not equally sensitive throughout the range of sensations. For example, in complete quiet, a dropped pin can be heard quite clearly, but against a background noise it becomes quite inaudible, although the pin must make the same noise in each case. The presence of the background noise reduces the sensitivity of the ear so that the noise of the pin passes unnoticed.

This effect was investigated by Weber and Fechner, who concluded that the loudness of a sound is approximately proportional to the logarithm of the intensity of the sound. In view of this, it would appear to be convenient to measure the intensity of sound on a logarithmic scale, for then equal increments in intensity on this scale would result in roughly equal increments in loudness.

If two sounds have intensities I_1 and I_2, then on this logarithmic scale the intensity of the first is said to be X *bels* higher than the second, where

$$X = \log_{10} \frac{I_1}{I_2} \qquad . \quad . \quad . \quad . \quad (32)$$

This only gives the value of I_1 relative to I_2, for if their ratio remains constant, X will take the same value whatever the actual values of I_1

and I_2. If I_2 is taken as some reference sound of known intensity, then the absolute value of I_1 can be found. The standard value chosen for I_2 is the intensity of the quietest sound which the ear can detect, called the *threshold of audibility*; this value varies for different observers, but the standard value is taken as 10^{-16} watts.cm^{-2}. The intensity of I_1 with respect to this standard intensity is then usually described as 'X bels above threshold'.

The bel is rather a large unit and so the *decibel* (abbreviated to db) is also used, where 1 bel is equal to 10 decibels.

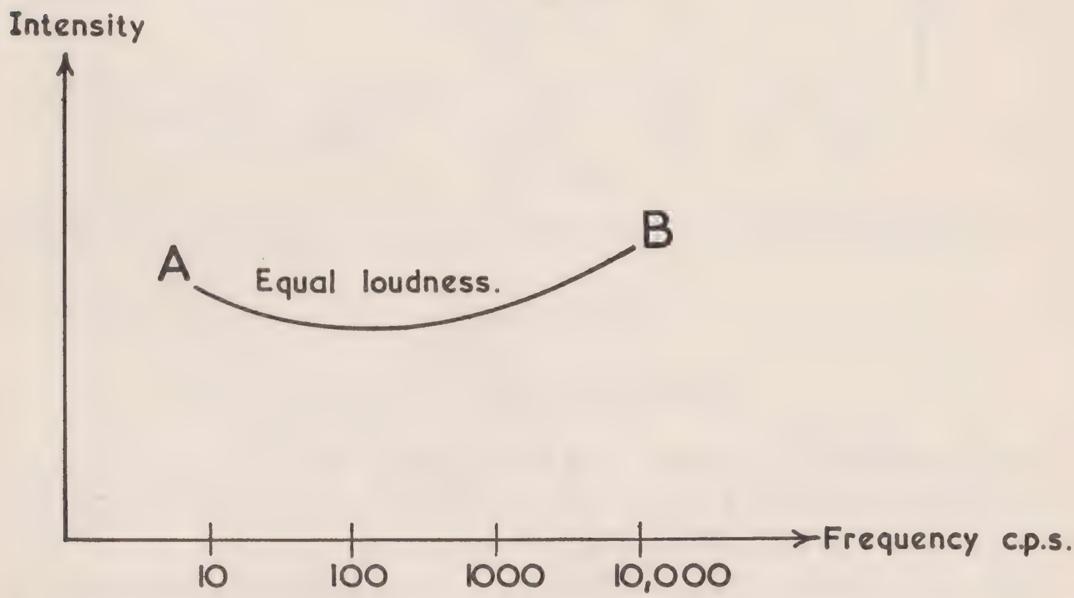


Fig. 10.21

Unfortunately a further complication arises because the ear is not equally sensitive to all frequencies; that is, it does not register sounds of the same intensity but different pitch with the same loudness. This is shown diagrammatically in Fig. 10.21, where the line AB is drawn through points representing notes of equal loudness as judged by an observer. It will be seen that the ear has its maximum sensitivity in the range 2000–3000 cps, outside this interval, say at either 10 cps or 10,000 cps, a sound of much higher intensity is needed to produce the same loudness. Because of this variation in sensitivity, the decibel scale of intensity does not approximate to an equal loudness scale *when applied to notes of different pitch.*

A further system is in use in this case, and is illustrated in Fig. 10.22. Intensity is plotted against frequency on logarithmic scales and the curve showing the threshold of audibility for an average observer is drawn in as a solid line. A standard frequency of 1000 cps is now chosen and points such as A, B, C are marked on the diagram representing 1000 cps sounds at intensities of 20, 40, 60 db above threshold;

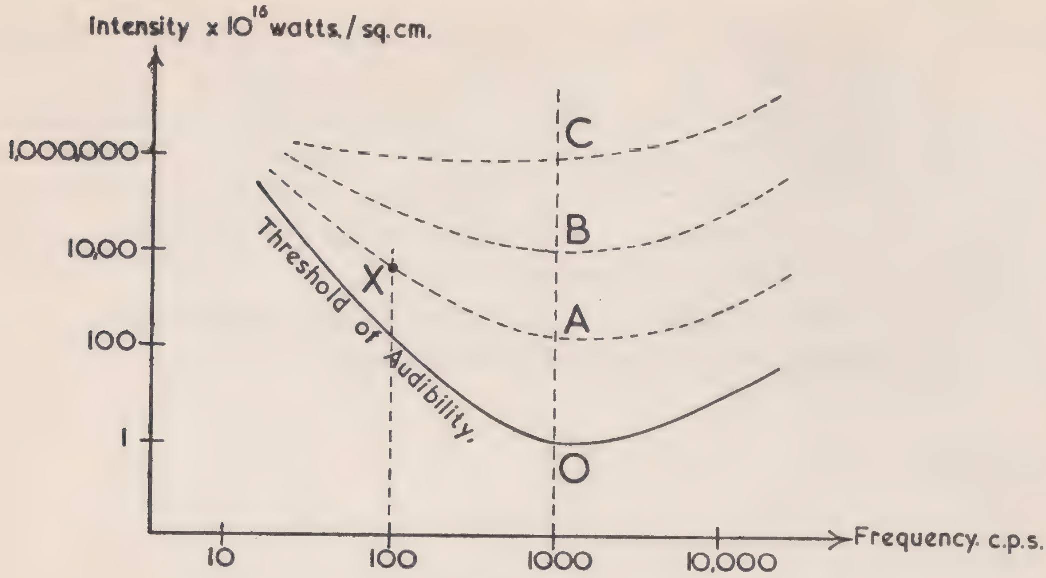


Fig. 10.22

this can be done without reference to an observer, for 20 db is 2 bels

$$\text{or } 2 = \log_{10} \frac{I_1}{I_2},$$

$$\text{hence } \frac{I_1}{I_2} = 100.$$

Thus the distance between A and O on the graph must represent an intensity ratio of 100 : 1, or must be equal to the distance between the 100 mark and the 1 mark on the intensity axis.

Next, an 'average observer' traces out equal loudness curves, trying to match the loudness of sounds at other frequencies with the standard 1000 cps tone at 20, 40, 60 db, etc. These curves are dotted in and marked as 20, 40, 60 on the diagram.

It will be seen then that at a point such as X on the diagram, a 100 cps note appears to an observer to be just as loud as the 1000 cps note represented by the point A; this has an intensity of 20 db above threshold, hence the 100 cps. note is said to have a *loudness* of 20 *phons*.

In other words, a note of any frequency whose loudness is X phons appears to an observer to be just as loud as a 1000 cps tone of intensity X db above threshold.

SUMMARY OF NEW UNITS INTRODUCED IN THIS CHAPTER

Quantity	*Dimensions*	*c.g.s. and derived units*	*f.p.s. and derived units*	*Gravitational units*
Intensity .	MT^{-3}	watt.cm^{-2}	Not in common use.	
Decibel .	PURE NUMBER			

EXERCISES 10

1. Show that when the pressure and volume of a gas are related by the equation $pv^n =$ constant, the elasticity of the gas is equal to np.

   Calculate the velocity of sound in air at $-80°$ C. if the density of air at 0° C. and 10^6 dynes per sq. cm. pressure is 1·275 gm. per litre and the ratio of the principal specific heats of air is 1·41.

   (London Univ. Inter. B.Sc.)

2. Give a brief account of any important and characteristic wave phenomena which occur in sound. Why are sound waves in air regarded as longitudinal and not transverse?

   An observer looking due north sees the flash of a gun 4 seconds before he records the arrival of the sound. If the temperature is 20° C. and the wind is blowing from east to west with a velocity of 30 miles per hour, calculate the distance between the observer and the gun. The velocity of sound in air at 0° C. is 1,100 ft. per sec. Why does the velocity of sound in air depend upon the temperature but not upon the pressure? (Northern Univ. H.S.C. Schol. level.)

3. How does the velocity of sound in a gas depend on temperature and pressure?

   The observer in an aeroplane flying horizontally at 240 m.p.h. releases a bomb and hears the sound of the explosion 20 sec. afterwards. Find the height of the aircraft, neglecting air resistance. (Velocity of sound in air = 1100 ft./sec.) (Oxford Univ. Schol.)

4. A ship travelling due north at 1 m./sec. in a thick fog fires a detonator in the sea alongside and receives an echo from a buoy on the port side 1·2 sec. later. Fifteen minutes later a repetition of the experiment yields the same result. What is the bearing and distance of the buoy? Describe the type of apparatus you would use to make these measurements. (Velocity of sound in sea water = 1500 m./sec.)

   (Oxford Univ. Schol.)

5. An observer standing close beside an anti-aircraft gun notices that the shell explodes 5 sec. after it has been fired. The sound of the explosion reaches him 9 sec. later. If the angle of elevation of the gun is 45°, calculate to within the nearest hundred feet the height at which the shell explodes. (Velocity of sound in air = 1110 ft./sec.; $g = 32$ ft./sec./sec.) (Cambridge Univ. Schol.)

6. Describe how you would determine the velocity of sound in air at room temperature, pointing out the precautions you would take in order to obtain an accurate value.

   A ship at sea sends out simultaneously a wireless signal above the water and a sound signal through the water, the temperature of the water being 4° C. These signals are received by two stations, A and B, 20 miles apart, the intervals between the arrival of the two signals being 13·2 sec. at A and 17·6 sec. at B. Find the bearing from A of the ship relative to AB. The velocity of sound in sea-water at $t°$ C. $= 4{,}756 + 11t$ ft.sec.$^{-1}$. (London Univ. G.C.E. Advanced level.)

7. Give details of *one* good method of determining the velocity of sound in the open air.

Write down an expression for the velocity of sound in a gas, and show (*a*) that the velocity is independent of the pressure of the gas, (*b*) that the velocity is proportional to the square root of the absolute temperature.

Explain the effect of a vertical temperature gradient on the propagation of sound. (Cambridge G.C.E. Advanced level.)

8. Give a brief account of the evidence in support of the view that sound is propagated as a wave motion through the air. What are the physical factors that determine the velocity of propagation of such waves?

Indicate the chief sources of error in measuring the velocity of sound in the open air and describe a good method of finding this velocity. (Northern Univ. G.C.E. Schol. level.)

9. Write down theoretical expressions for the velocity of longitudinal waves (i) through a gas, (ii) along a thin rod, defining the symbols used.

Calculate the velocity of sound in argon at $-100°$ C.

Assume the following values: Ratio of principal specific heats of argon $= 5/3$; density of argon at S.T.P. $= 1{\cdot}98 \times 10^{-3}$ gm./cm.^{3}; density of mercury $= 13{\cdot}6$ gm./cm.^{3}; $g = 981$ cm./sec.^{2}

(London Univ. Inter. B.Sc.)

10. Explain the factors which determine the velocity of longitudinal waves in (*a*) a gas, (*b*) a solid. Describe how the velocity of sound in a brass rod may be compared with that in air.

If the velocity in a brass rod is 12,000 ft. per sec. and its density is 530 lb. per c.ft., find the value of Young's modulus for the metal in tons wt. per sq.in. (London Univ. Inter. B.Sc.)

11. What is the Doppler effect? What quantitative deductions about the velocity of a body can be made by means of it?

The wavelength of a spectrum line emitted from a star is found to be 6562·912 A. instead of the normal value, 6562·784 A. What deductions can be made about the velocity of the star, and what would be the observed wavelength of a line (emitted by the same star) of which the normal value is 4861·327 A.? (Speed of light $= 3 \times 10^{10}$ cm./sec.)

(Cambridge G.C.E. Advanced level.)

12. Derive expressions showing how the apparent frequency of a note heard by an observer is affected by (*a*) motion of the source, (*b*) motion of the observer, in each instance the motion being along the line of propagation of the sound.

A motor-car is fitted with twin horns differing in frequency by 256 vibrations per second. Calculate the difference of frequencies of the notes heard by an observer when the car, sounding the horns, is

approaching him at 40 m.p.h. Take the velocity of sound in air as 1,120 ft. per sec. (Northern Univ. H.S.C.)

13. Describe a method for the determination of the velocity of sound in free air. Explain how this velocity is affected by changes in atmospheric conditions.

A train, sounding a whistle of natural frequency 500 vibrations per sec., approaches a stationary observer at a speed of 72 km. per hour. Find the change in the frequency of the note heard by the observer as the train passes him. Derive any formula employed. (Assume that the velocity of sound in air is 340 metres per sec.)

(Northern Univ. G.C.E. Advanced level.)

14. What do you understand by the Doppler effect? A whistle of 1000 cycles/sec. pitch is attached to one end of a light tube 2 ft. long so that it can be sounded while the tube is rotating freely in a vertical plane about a horizontal axis through the other end. If the velocity of the whistle when the tube is horizontal is 16 ft./sec., find the upper and lower limits of the pitch of the sound heard by an observer on the ground viewing the motion end-on. (Velocity of sound = 1100 ft./sec.)

(Oxford Univ. Schol.)

15. What is the Doppler effect? Explain how it can be used to measure one component of the velocity of a body, by means of sound waves and light waves. How has the Doppler effect been used to prove that Saturn's ring is composed of discrete particles? (Cambridge H.S.C.)

16. A vibrating tuning-fork is moving steadily with a velocity of 150 cm./sec. normally towards a wall from which the sound waves are reflected. If the frequency of the fork is 512 sec.^{-1}, what will be the frequency of the beats heard by a stationary observer who has just been passed by the fork? (Velocity of sound = 330 m./sec.)

(Cambridge Univ. Schol.)

17. A train is moving with uniform velocity v on a straight track between two bridges A and B over the track, the motion being towards A. An observer on the train hears the echo of the train's whistle reflected from each of the bridges. If the velocity of sound is V, find the ratio of the wavelengths of the waves reflected from A and B and the ratio of the frequencies of the echoes heard by the observer.

(Cambridge Univ. Schol.)

18. Explain the origin of the beats heard when two tuning-forks of slightly different frequencies are sounded together. Show that it is only possible to decide which fork has the higher frequency if the frequency of the beats is *decreased* when one fork is loaded with a small piece of wax.

A simple pendulum, set up to swing in front of a clock pendulum of time period 2 sec., is observed to gain so that the two swing in phase at intervals of 18 sec. Calculate (a) the time period and (b) the length of the simple pendulum.

(Northern Univ. G.C.E. Advanced level.)

19. Two tuning-forks of frequencies 256 and 257·2 vibrations per second are sounded together. Describe and explain what is observed, stating clearly the principle involved.

How can this principle be made use of in determining the period of a simple pendulum? Examine carefully the errors involved in such a determination.

(Cambridge Univ. Schol., Girton and Newnham Colleges.)

CHAPTER 11

STATIONARY WAVES

11.1 The Reflection of Sound

(a) Echoes

Everyone has heard at some time or other an echo; a loud sound occurs and is then repeated more quietly a short time later. Echoes are most commonly heard when one is standing at some distance from a large flat vertical surface such as a cliff, the front of a large building, the trees at the edge of a wood, or the far side of a deep valley. An observer hears first of all an original sound direct from some nearby source; subsequently, the same sound is heard after it has travelled to the distant surface, been 'reflected' by it and therefore travelled back to the ear of the observer. All forms of wave motion can be reflected by a suitable obstacle placed in their path. A familiar case of the reflection of waves is that which occurs in water waves, a motor-boat passing along a river causes a bow wave which produces a wash when it arrives at the bank, but a few seconds after the boat has passed, a second wash occurs. This is due to the bow wave which spread out from the boat to the opposite bank, was reflected there and crossed back again, arriving a few seconds after the original wash (see Fig. 11.1). It is quite easy to see this second set of waves coming back across the river, in addition to experiencing the wash which they cause at the bank.

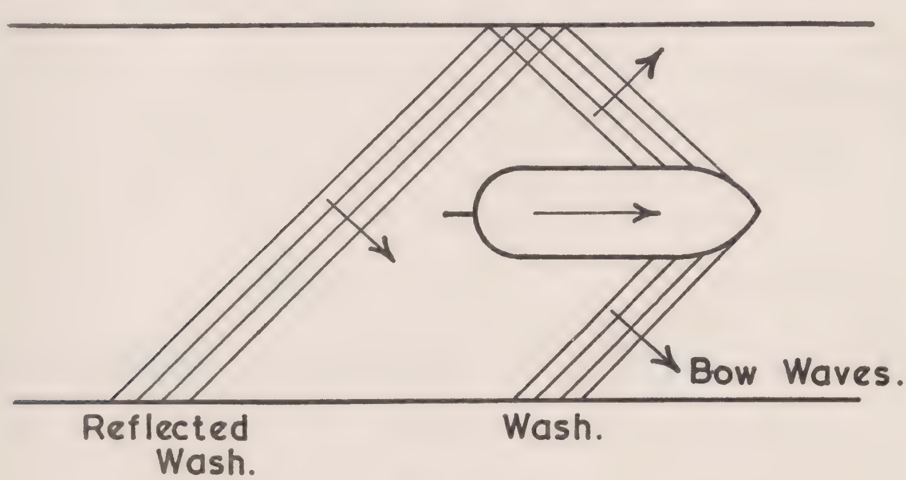


Fig. 11.1

The student may wonder how something as rough as a cliff face or the trees at the edge of a wood can be described as a reflector, when a highly polished surface is needed to reflect light. To be a good reflector,

a surface has to treat all parts of an advancing wave front similarly; this necessitates a surface whose flatness does not depart from plane by more than a fraction of the wavelength of the wave motion. Now light has a wavelength of about 0·0006 mm, hence a highly polished surface is needed, but the sounds of speech have a wavelength of about 10 feet, thus a surface may depart from the flat by several feet and still behave as though it were 'highly polished' to a sound wave.

(b) Reflection of a Train of Sound Waves

We have seen that a large surface reflects a sound back to its source. This must mean that the wavetrain representing the original sound has its direction of travel reversed on meeting the obstacle; further, since an echo seems to be just the same as the original sound, it would appear that the returning wavetrain is exactly the same as the original. It would be quite possible to examine the physics of a sound wave meeting an obstacle and to show that, within certain limits, the above statements are true; the work, however, is not easy and is better postponed until later.

If the statements made above are accepted, they indicate that a continuous train of waves meeting an obstacle and being reflected will give rise to two identical trains of waves passing through the same medium but in opposite directions. These two trains will interact with each other, producing some results which are of great importance in the study of sound.

11.2 Interaction of Two Wavetrains

Initially, let us consider any travelling waveform whose *particle displacement* can be represented by a graph such as Fig. 11.2. It was

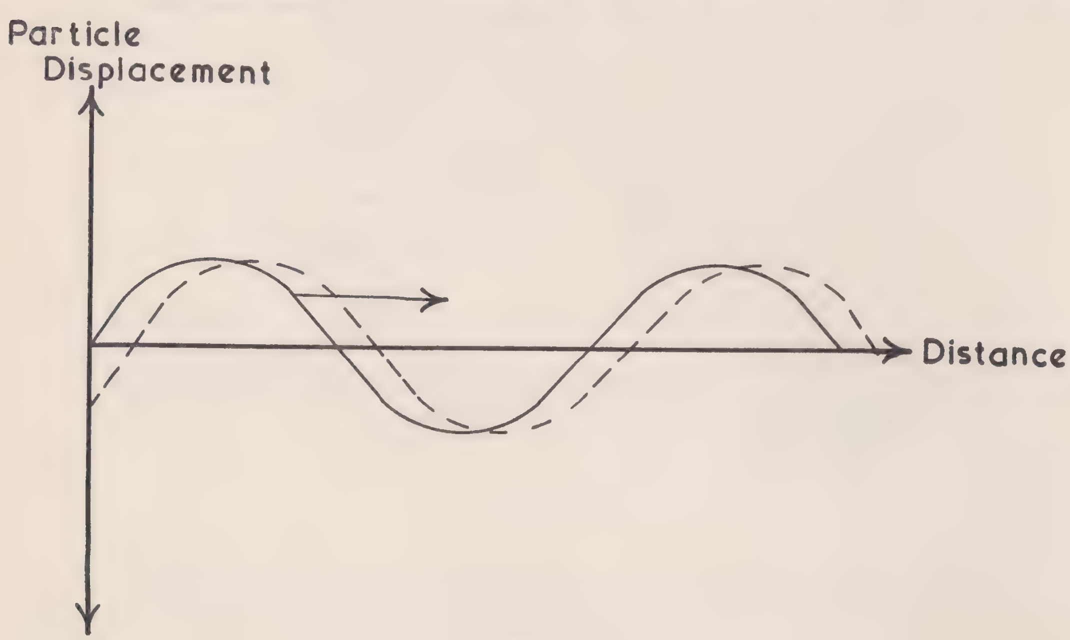


Fig. 11.2

seen on page 283 that both a longitudinal and a transverse wave motion at any instant can be depicted by a graph such as the solid curve. If the wave is moving to the right, then after a brief interval of time the particle displacement will be shown by the dotted curve.

If two identical wavetrains are moving through the same region at once at equal speeds but in opposite directions (i.e. a sound wave and the reflected wave), then at some instant they can be represented by the curves of Fig. 11.3 (*a*), where the arrows show the direction of motion of each waveform. An instant later they will be as shown in Fig. 11.3 (*b*), the two crests A and B having moved closer together, and at some later instant the two crests will coincide as in Fig. 11.3 (*c*).

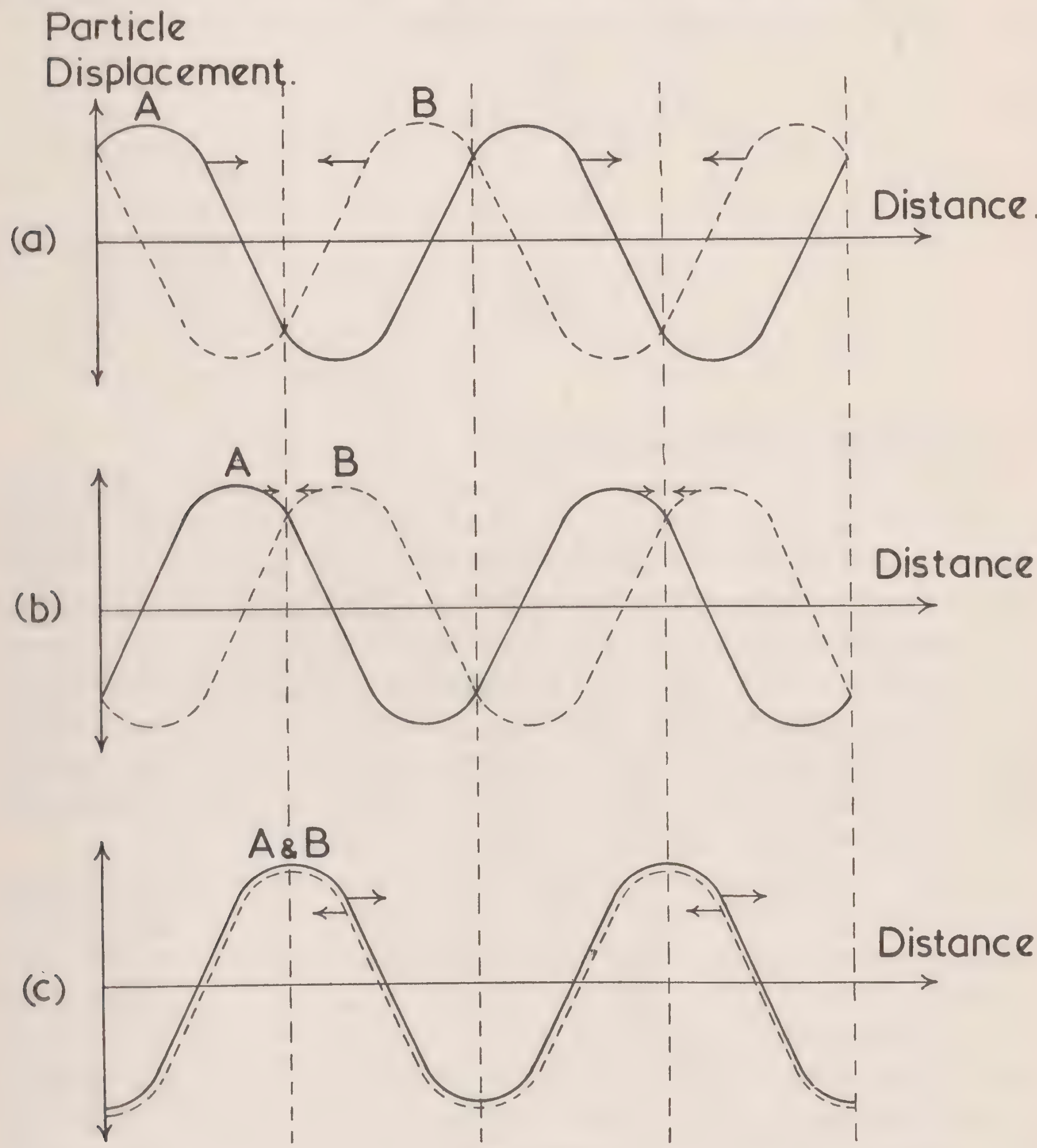


Fig. 11.3

In this graph the particle displacement due to each waveform is shown as a separate curve. Obviously if two waveforms disturb the particles of the medium through which they are travelling, then the

actual displacement of any particle will be the sum of the displacement it receives due to each wave alone. Thus the total displacement of any particle represented in Fig. 11.3 can be found by adding together the ordinates of the two curves as in Fig. 11.4.

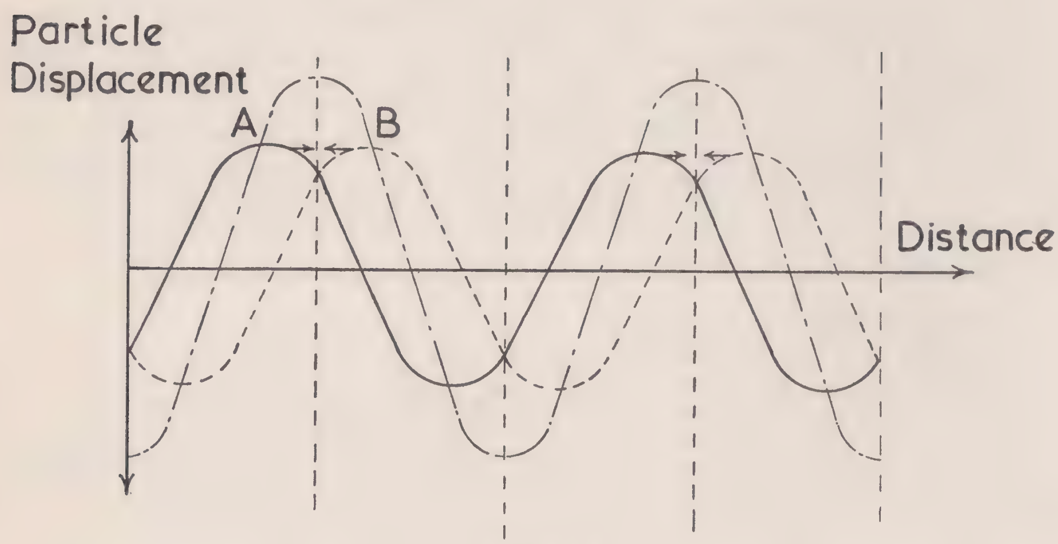


Fig. 11.4

11.3 Formation of Standing Waves

To find the total effect of two waves passing through the same medium, let us start from the condition represented in Fig. 11.3 (*c*) where the two waveforms just coincide, this is repeated on a different scale as Fig. 11.5 (*a*); adding the ordinates at any point (i.e. adding *OP* to *OQ* and plotting it as *OR*) gives the resultant particle displacement at this instant of time, shown in the figure as a chain-line. Now repeat the process at some slightly later time, as in Fig. 11.5 (*b*). Each waveform will move on a small distance but in opposite directions, and the resultant particle displacement is again found by adding the ordinates of the two waveforms, resulting in the chain-line curve.

It will be noticed that this curve passes through zero at points *A* and *B* and that these points are in the same position as the zeros of the combined curve shown in Fig. 11.5 (*a*). This must be so owing to the symmetry of the two moving waveforms, for if they are both moved from the coincident position of Fig. 11.5 (*a*) through equal distances in opposite directions, then they bring to the point *A* equal and opposite ordinates (shown in more detail in Fig. 11.6). These two ordinates must always cancel out at this point, however far the waveforms are moved.

An important point thus emerges—as these two waveforms move, they will always combine to produce zero displacement at points such as *A* and *B* (and corresponding points in subsequent cycles). Moreover,

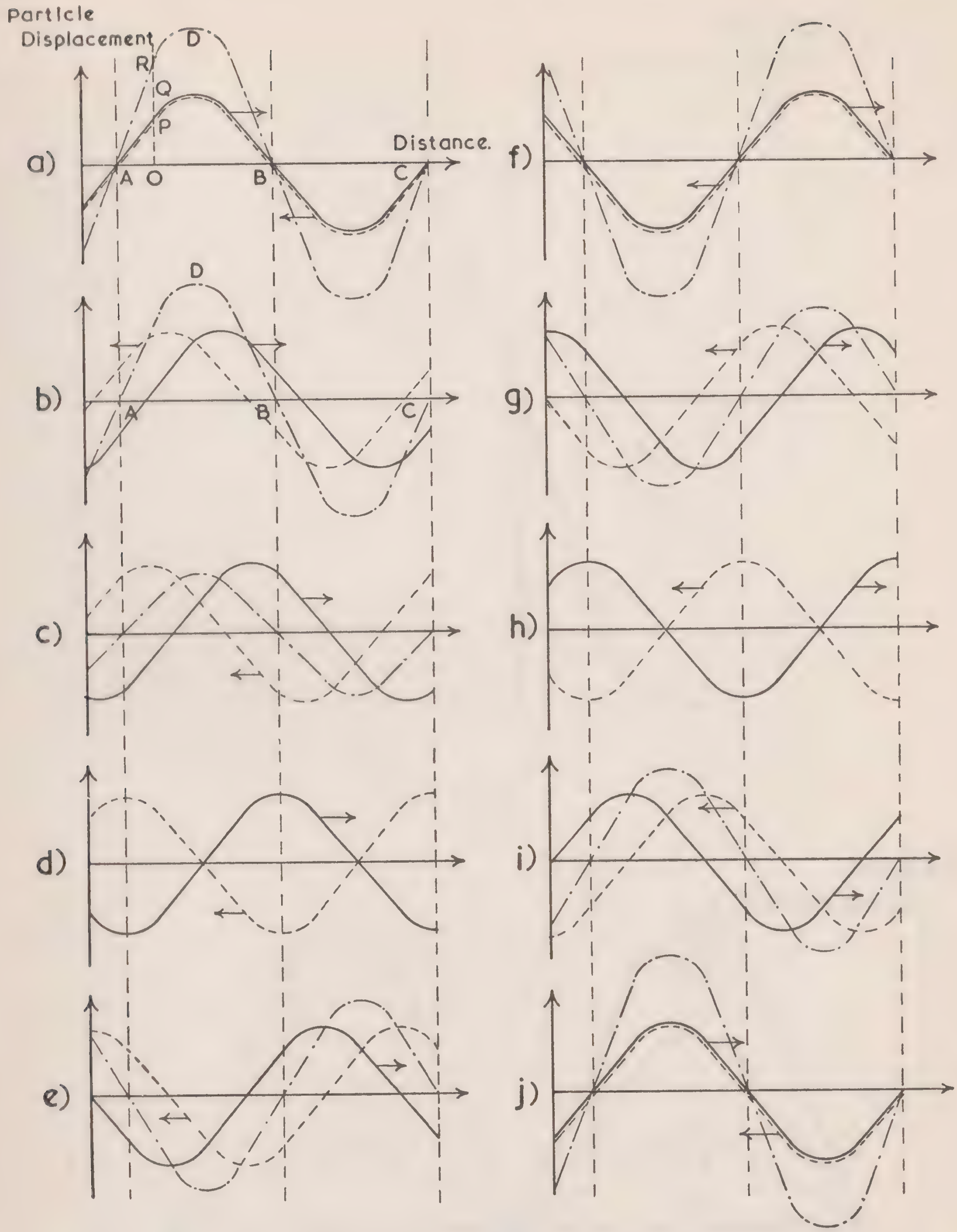


Fig. 11.5

these points are *fixed in space*, i.e. they do not move along, although they are produced by the interaction of two travelling waves. Thus, whatever the resultant waveform may be, it is contained within fixed points and does not move along, it is a *stationary* or *standing* waveform.

The actual nature of the waveform is revealed in Figs. 11.5 (*a*) to (*j*) and for clarity, the resultant waveforms from Fig. 11.5 are collected together and superimposed in Fig. 11.7, and it will be seen that in between the points of no disturbance A, B and C, the particle displace-

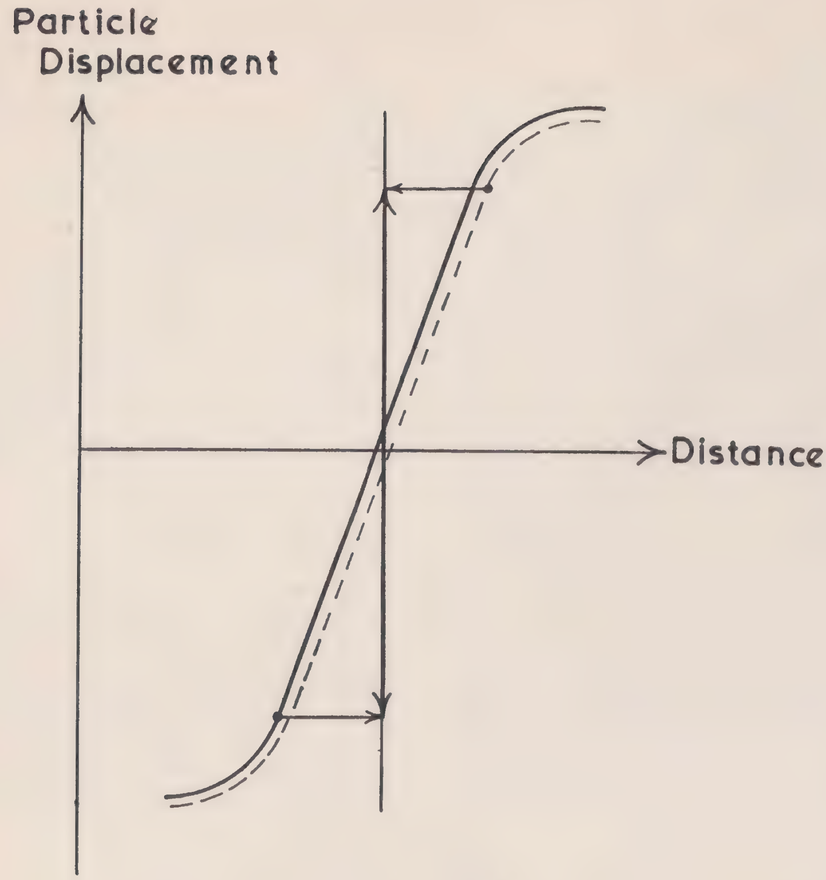


Fig. 11.6

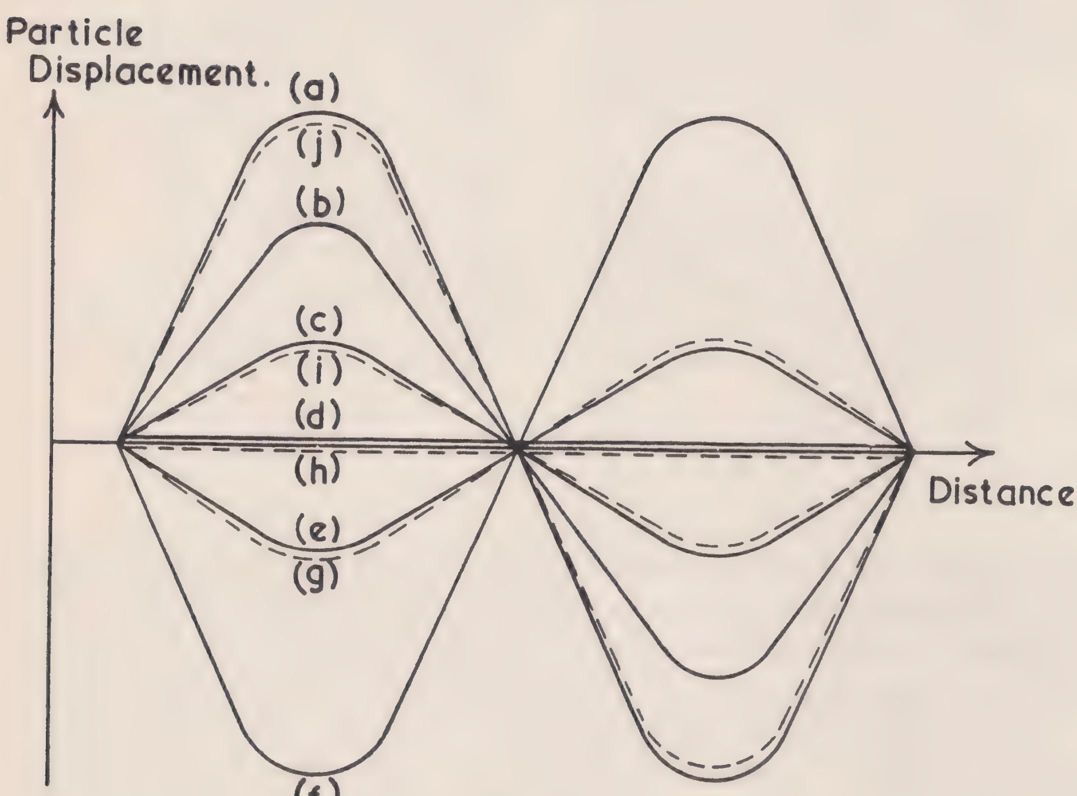


Fig. 11.7

ment begins at a maximum in one direction (*a*), sinks to zero (*b*, *c*, *d*), increases to a maximum in the other direction (*e*, *f*), returning once more to zero (*g*, *h*) and then goes back to its original maximum value (*i*, *j*).

Points such as A, B and C, where no particle displacement occurs, are called *Nodes*, while the points half-way between, where the particles suffer maximum displacement, are called *Antinodes*. From the diagrams of Fig. 11.5, it will be seen that the wavelength of the standing wave is exactly the same as that of either of the travelling waves; moreover, one cycle of particle displacement in the standing wave occurs while one cycle of the travelling wave passes, thus the frequency of the particle motion in both the standing wave and travelling waves are the same. This simplifies experimental work on travelling waves considerably; it is very difficult to measure the wavelength of a travelling wave, but if it is used to form a set of standing waves, then the wavelength can be measured directly.

The phenomenon described above can be examined mathematically as follows. The particle displacement of a wave motion of amplitude A, periodic time τ and wavelength λ, moving in the direction of increasing values of x (i.e. to the right in conventional cartesian co-ordinates), was shown on page 279 to be represented by

$$y_1 = A \sin 2\pi\left(\frac{t}{\tau} - \frac{x}{\lambda}\right).$$

A similar wave moving in the direction of decreasing values of x, that is, to the left, is represented by:

$$y_2 = A \sin 2\pi\left(\frac{t}{\tau} + \frac{x}{\lambda}\right).$$

If these two interact, the total particle displacement y is given by:

$$\begin{aligned} y &= y_1 + y_2 \\ &= A\left[\sin 2\pi\left(\frac{t}{\tau} - \frac{x}{\lambda}\right) + \sin 2\pi\left(\frac{t}{\tau} + \frac{x}{\lambda}\right)\right] \\ &= A\left(\sin\frac{2\pi t}{\tau}\cos\frac{2\pi x}{\lambda} - \cos\frac{2\pi t}{\tau}\sin\frac{2\pi x}{\lambda}\right. \\ &\qquad \left. + \sin\frac{2\pi t}{\tau}\cos\frac{2\pi x}{\lambda} + \cos\frac{2\pi t}{\tau}\sin\frac{2\pi x}{\lambda}\right) \end{aligned}$$

$$\text{or } y = 2A \sin\frac{2\pi t}{\tau}\cos\frac{2\pi x}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

From this equation it is seen that if

$$x = \lambda/4,\ 3\lambda/4,\ 5\lambda/4, \text{ etc.} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

then the factor $\cos(2\pi x/\lambda)$ is zero, and thus the displacement at each of these points is zero whatever the value of the time t in the factor $\sin(2\pi t/\tau)$.

Equation (2) indicates that zero displacement, i.e. the node, occurs at fixed positions in space, and thus Equation (1) represents a waveform which is also fixed in space, in fact it is a stationary or standing waveform. Further, from Equation (2) the distance between any pair of adjacent nodes is $\lambda/2$ and thus the wavelength of the standing wave is λ, i.e. the same as that of the travelling wave.

At any fixed position x' other than one of the nodes, the factor $\cos(2\pi x'/\lambda)$ takes some constant value, say k; thus

$$y = 2kA \sin \frac{2\pi t}{\tau}.$$

At this position, the particle displacement is simple harmonic (due to the factor $\sin(2\pi t/\tau)$), and has the same period τ as the travelling waveforms. The amplitude of the particle displacement at this point is $2kA$, and this takes its maximum value when k is a maximum, i.e. when $\cos\frac{2\pi x}{\lambda}$ is a maximum. The cosine takes its maximum value $+1$ whenever the angle is $0, 2\pi, 4\pi$, etc., and its maximum negative value -1 at $\pi, 3\pi$, etc.

These will give the same value for the maximum amplitude, the negative sign merely indicating that in adjacent half-cycles of the standing wave, the particles are moving in the opposite direction at the same instant.

$$\text{Thus if } \frac{2\pi x}{\lambda} = 0, \pi, 2\pi, \text{ etc.,}$$

$$\text{then } x = 0, \lambda/2, \lambda, \text{ etc.} \qquad (3)$$

and the points of maximum disturbance (i.e. the antinodes) occur half-way between the nodes indicated by Equation (2).

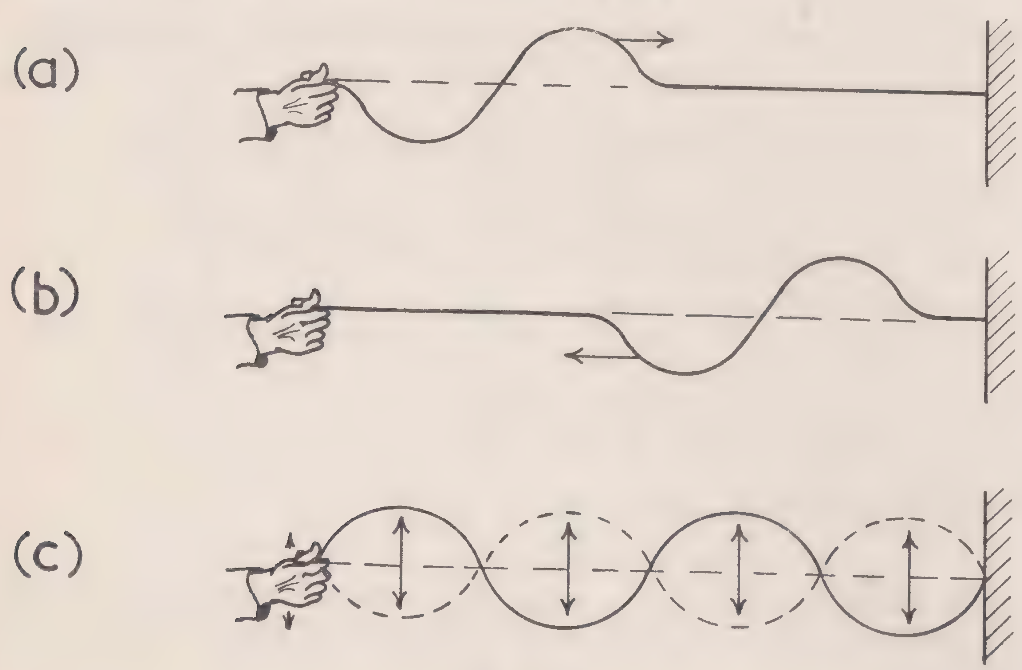


Fig. 11.8

Standing waves are very often seen in everyday life; for example, if a cord is tied to a support at one end and the other end given a quick flick up and down, a wave is seen to run along the cord and is reflected (with the waveform inverted) from the fixed end as shown in Fig. 11.8 (*a*) and (*b*). But if the end is moved up and down continuously at a certain frequency so that a train of waves runs down the cord and is reflected from the far end, then standing waves are set up; as shown in Fig. 11.8 (*c*).

11.4 Reflection of Waves at Fixed or 'Free' Boundaries

(a) Fixed Boundaries

If a travelling wave runs along a cord fixed at one end as in Fig. 11.8, the wave is reflected from the far end, and the two sets of waves combine to form standing waves on the string. Now we know that this standing wave has nodes spaced at half wavelength intervals, but where do they occur on the string? The position of one node can be identified straight away; the far end of the string is rigidly fixed, hence it *must* be a node, and all the other nodes will be spread along the string at half wavelength intervals from the fixed end as shown in Fig. 11.8 (*c*).

This fact also enables us to examine the reflected waveform, for if the fixed end is to be a node, the incident and reflected waves must produce equal but opposite displacements at this point and so cancel out;

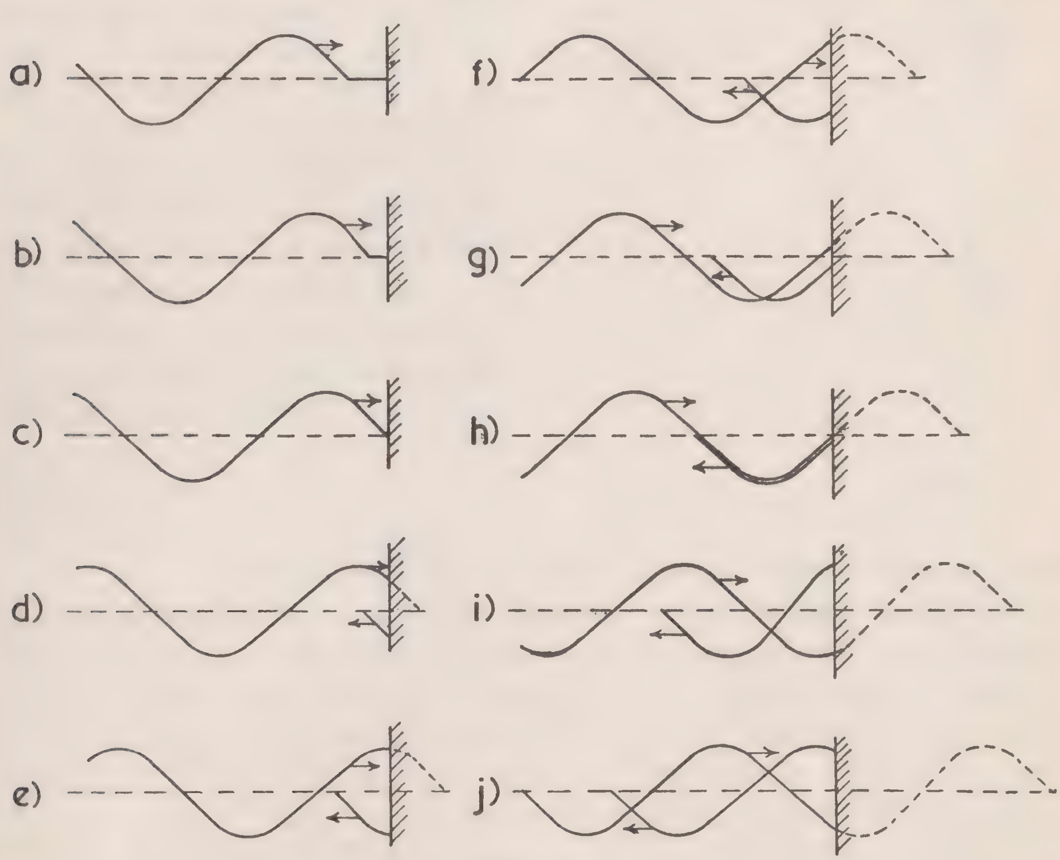


Fig. 11.9

the progress of a wave up to such an obstacle and the reflection from it is shown in more detail in Fig. 11.9.

It will be seen that the reflected wave not only goes in the opposite direction but is 'upside down', the positive half-cycles are reflected as negative ones. This is equivalent to changing the phase of the reflected wave by half a cycle; thus, *at a rigid boundary,* the particle displacement waveform suffers a phase change of half a cycle on reflection.

(b) Free Boundary

The possibility of a wave being reflected from a fixed end of a string is easily seen, but it is not so easy to convince oneself that a similar reflection will take place from a free end. If it is remembered, however, that a wave motion carries energy with it, then when the wave motion comes to the end of the medium in which it is travelling—whether in this case that 'end' is a fixed point in the string, beyond which the waveform cannot go, or just a free end of the string—then something must happen to the energy carried by the wave motion. We have no evidence that the energy merely accumulates at the end of the string, hence it seems reasonable to suppose that energy is carried away again by a reflected wave travelling in the opposite direction. The presence of reflected waves can readily be demonstrated in a string with a fixed end, but the experiment is rather difficult when the string has a free end. The theory of waves in a string developed in the previous chapter demanded a string having no stiffness, but subjected to a constant tension, and obviously a string with one end free cannot be subjected to a uniform tension. If, however, a string is allowed to hang vertically with the lower end free, the weight of the string itself provides a tension which, although it varies along the length of the string, is good enough for the purpose of demonstration.

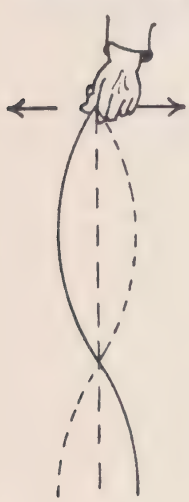


Fig. 11.10

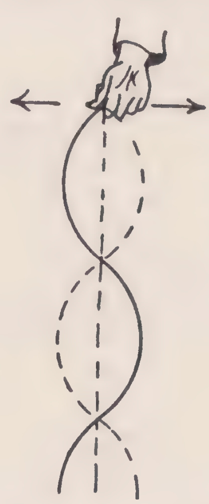


Fig. 11.11

If a piece of rubber tube is loaded with sand, and allowed to hang vertically, and the upper end moved to and fro, then standing waves, as shown in Fig. 11.10 or 11.11, will be set up. Now standing waves can appear only as a result of two similar waves moving in opposite directions, thus this experiment indicates that a reflection must have occurred at the free end of the rubber cord.

It will be noticed that an antinode occurs at the free end, thus at this point the incident and reflected waves are assisting each other,

both producing a displacement in the same direction. The reflected wave is shown in Fig. 11.12 which should be compared with the case of a fixed end, shown in Fig. 11.9. *At a free boundary* the particle displacement waveform suffers no change of phase on reflection.

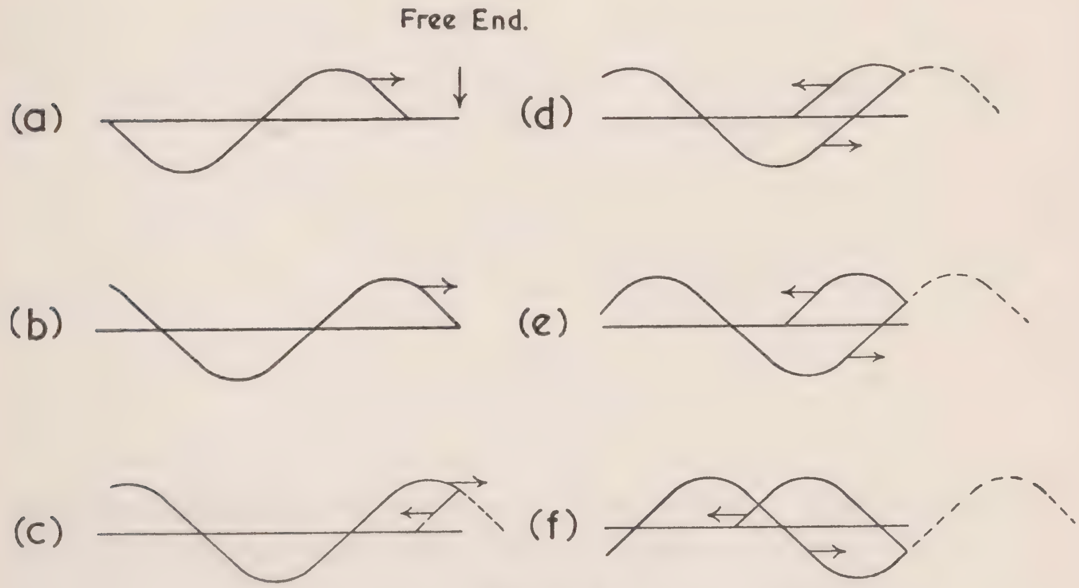


Fig. 11.12

11.5 Resonance

So far we have discussed the reflection of a displacement waveform at the end of a cord. We must now turn our attention to the reflected wave and see what happens to it when it gets back to the start of the cord.

The start of the cord will usually be fixed, for example Fig. 11.9 shows it held in a hand, but it may be free. In either case, however, it constitutes a boundary at which reflection must take place, just as described in the preceding paragraph. Thus, when the reflected wave returns to the start of the cord, it will be reflected in turn and will begin a second run down the cord. On this second run, it combines with the new waves being fed into the cord by the source.

This reflected wave may be in step with the next wave from the source, in this case they will reinforce each other and build up a very large wave; but they may also be out of step, in which case they will cancel each other out. Either of these effects may happen partially or completely according to the actual phase relation between the two waves. The phase relation in turn depends on the length of the cord, for this will determine the delay experienced by the reflected wave in running down the cord and back again. It should be possible therefore to calculate the length of the cord which will bring the reflected wave and the wave from the source into phase and so build up large waves in the cord.

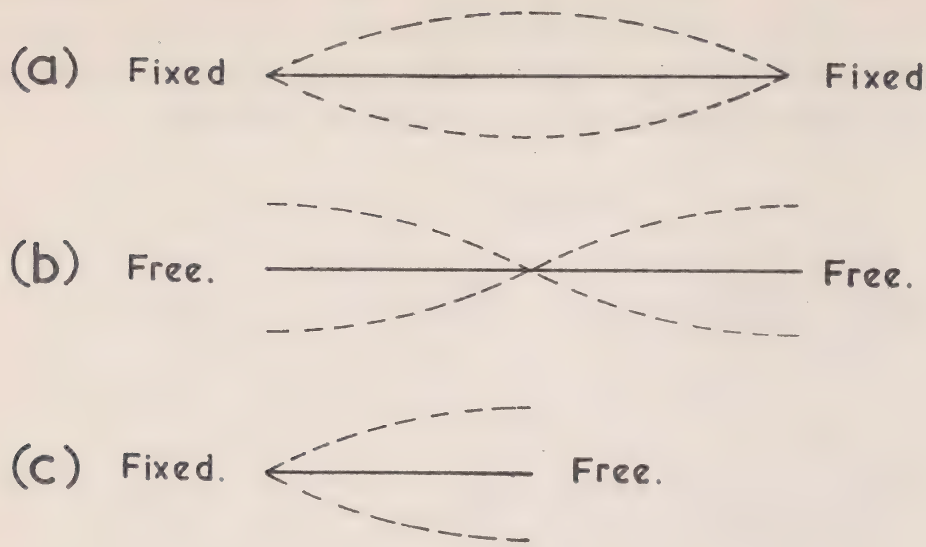


Fig. 11.13

There is, however, another condition which must be met if standing waves are to exist on a cord, and if fulfilled this will lead us quite simply to the solution of the problem above.

Since each end of the cord must be either fixed or free, *each* end of the cord must be either a node or an antinode; the length of the cord must thus be such that the standing wave can 'fit' on the cord and satisfy the necessary end conditions. The lengths of the cord which will satisfy these conditions for a fixed wavelength of standing wave (and all possible combinations of fixed or free ends) is shown in Fig. 11.13. Some of these terminations are not possible for the stretched cord referred to above, but apply to other vibratory systems to be described later.

Consider now the case shown in Fig. 11.13 (*a*), i.e. a cord *half a wavelength long* and fixed at both ends; let us trace out the passage of a travelling wave in the cord. The source initiates a travelling wave and this takes a time equal to half its period to travel to the far end of the string. On reflection at the fixed end the wave suffers a phase reversal and then travels back to the start of the string, taking a time equal to a further half period. On reflection at the fixed point at the start of the string the wave suffers a further phase reversal. Thus at the start of its second run down the string, the reflected wave has suffered two phase reversals, which brings it back into phase, and has taken a time equal to one whole period; this brings it exactly into step with the next wave being initiated on the string by the source. The reflected wave thus adds to the next cycle; this is repeated in all subsequent cycles, until a large set of travelling waves is built up in the cord, and these combine, forming a large standing wave. In the argument above, it was explained that only a string of certain length would respond to a given

frequency and build up standing waves of large amplitude. Alternatively only some specific frequency can be used to produce standing waves of large amplitude in a string of given length. This property is called *Resonance* and the string is said to resonate at the frequency of the source.

Resonance occurs between any source of vibration and system capable of vibrating if the natural frequency of the latter is the same as the source. If the system is set in vibration by the source, it will perform one cycle of its motion and begin its second cycle just as it receives the second input cycle from the source. A relatively small input of energy from the source can thereby be made to build up a large vibration. Galileo observed that a massive pendulum could be set swinging by a series of puffs of air, so timed that they coincided with the natural swing of the pendulum.

Many examples of resonance will come to mind. A child on a swing can build up a large amplitude oscillation by moving his body to and fro through a very small distance, but timing his movements to coincide with those of the swing. Old cars have a tendency to produce loud rattles in the bodywork when the engine attains a certain speed, or a picture standing on the piano chatters when a certain note is struck.

The avoidance of resonance has also given rise to the popular impression that soldiers break step when marching over a cantilever bridge in case the beat of their feet should coincide with the natural frequency of vibration of the bridge; the beat might build up a large standing wave, and so destroy the structure.

11.6 Forced Vibrations

Although it has been explained above that a vibratory system such as a stretched string can carry a large standing wave only at its resonant frequency, it is quite evident that the string can vibrate at other frequencies if it is excited vigorously enough. However, the vibration produced in the string is of only small amplitude, it is not a standing wave, but a confusion of travelling waves and their reflections which are not in step with each other and so produce no general reinforcement of each other. The string is then said to be subjected to a *forced vibration*.

The magnitude of the forced vibration depends on two main factors. The difference between the forcing frequency and the resonant frequency of the system and also the *damping* of the vibration. Damping means the dying away of a vibration as energy is withdrawn from it. For example, a travelling wave in a stretched string carries energy with it, but the disturbance in the cord sets up some small sound waves in the surrounding air which carry energy away. The diminishing energy in the waveform in the string manifests itself as a gradual decrease in the amplitude of the travelling wave. The vibration dies away when all its energy has been withdrawn and the waveform has then been damped

out. If energy is withdrawn rapidly the system is said to be heavily damped and the motion dies away quickly. If the energy is removed slowly, the system is lightly damped and the vibration persists for a long time.

Consider now the response of a stretched string to various frequencies when subjected to different degrees of damping. Firstly, a

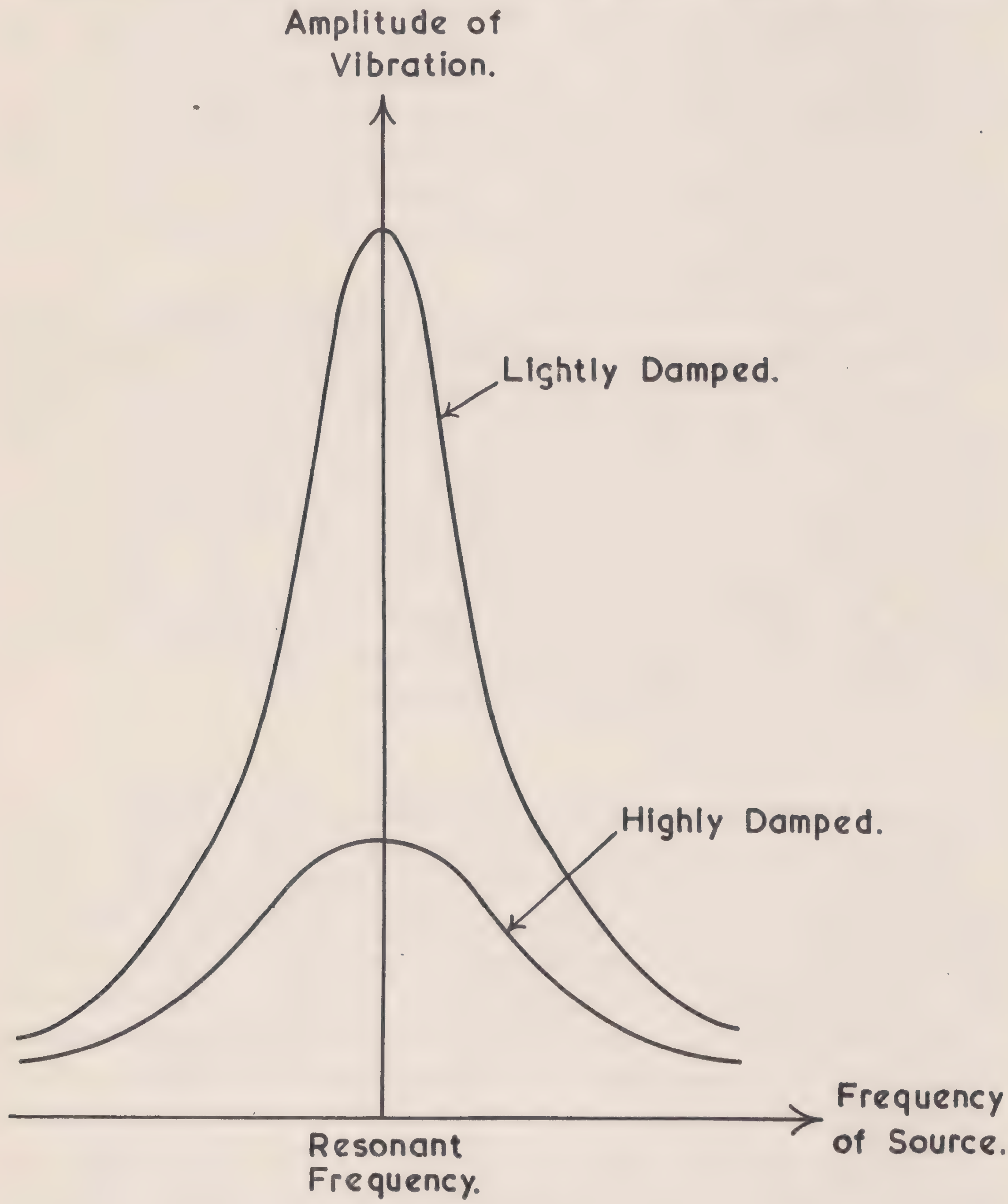


Fig. 11.14

system at its resonant frequency, we have seen that resonance is caused by travelling waves in the string being in phase with the subsequent vibrations from the source. If the damping is light, these waves will run up and down the cord for many cycles and the standing wave will be large, due to the summation of a great number of waves. If, however,

the damping is heavy, the travelling wave will last for a few cycles only, consequently there will be only a few cycles to combine into the standing wave, which will therefore be of reduced amplitude.

Next consider the same cases, but at a frequency slightly off resonance; suppose that the forcing frequency is such that the travelling wave, on its second run down the string, is one-fiftieth of a cycle out of step with the next wave in from the source. Then in its subsequent runs it will be 2/50, 3/50, etc., out of step until it reaches 25/50 or half a cycle on its twenty-sixth run. Now, if the damping is very slight, the amplitude of this wave will be not much reduced, thus it will very nearly cancel out the next cycle in from the source since it is exactly out of phase. This will happen for all subsequent waves and the standing wave produced in the string will be much reduced in amplitude. Suppose now that the damping is very heavy and that the travelling wave is damped out after about 10 cycles. The waves which will be left in the string to form a standing wave will be 1/50, 2/50, etc., down to 10/50 of a cycle out of phase with the next wave in from the source. The addition is not so large as if they were all exactly in step, but there is no cancellation as in the lightly damped system. The nett result is that the off-resonance response is not reduced by anything like the same extent in the highly damped system as in the lightly damped one. The effect is shown in Fig. 11.14, the two curves illustrate the amplitude of vibration which would build up in a lightly damped and a highly damped system, when excited by the same source at various frequencies, including the resonant frequency.

It will be seen that the lightly damped system responds very well at the resonant frequency and to a narrow band of frequencies on either side—it is described as 'sharply resonant' or very 'selective'. The highly damped circuit does not respond so well at the resonant frequency and embraces a wide band on either side, it is in fact not very selective in its frequency response.

11.7 Modes of Vibration of a Stretched String

A stretched string must be held rigidly at both ends, hence these two points must be nodes. One possible mode of vibration was shown in Fig. 11.13 (*a*), but in addition the string can take up any mode of vibration which places nodes at the ends; some of these modes of vibration are shown in Fig. 11.15. It will be seen that the string vibrates as either 1, 2, 3, etc., half-wavelengths of standing wave, the case shown in Fig. 11.15 (*a*) is called the fundamental mode—this was the condition considered in Section 11.5.

It was shown earlier (page 319) that the wavelength of a standing wave is the same as that of the two travelling waves producing it; thus,

taking the case shown in Fig. 11.15 (*a*), the wavelength of the travelling wave is given by:

$$\lambda_1/2 = l,$$

where l is the total length of the string;

$$\text{hence } \lambda_1 = 2l \quad . \quad . \quad . \quad . \quad (4)$$

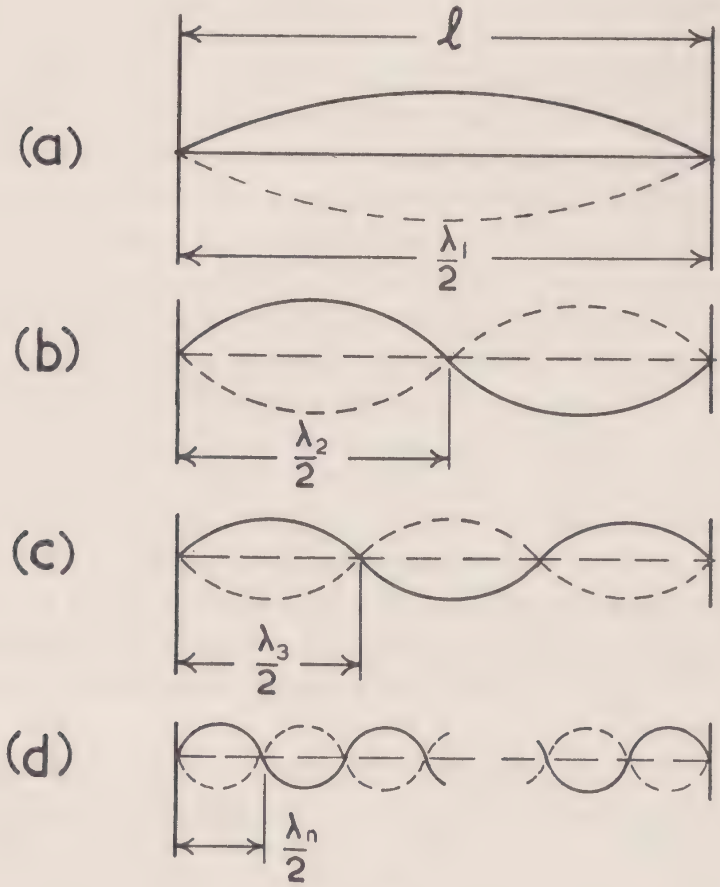


Fig. 11.15

But the wavelength of a travelling wave is related to its frequency f_1 and velocity v by:

$$v = f_1\lambda_1 \text{ (see Equation (6), Chapter 10).}$$

Substituting for λ_1 in Equation (4) gives:

$$2l = v/f_1$$
$$\text{or } f_1 = v/2l.$$

Further, it was shown on page 290 that the velocity of a travelling wave in a string is given by

$$v = \sqrt{\frac{T}{m_0}},$$

where T is the tension in the string and m_0 is the mass per unit length of the string.

Thus
$$f_1 = \frac{1}{2l}\sqrt{\frac{T}{m_0}}. \quad . \quad . \quad . \quad . \quad (5)$$

In the case shown in Fig. 11.10 (*b*) we have
$$2\left(\frac{\lambda_2}{2}\right) = l.$$

From case (*c*)
$$3\left(\frac{\lambda_3}{2}\right) = l,$$

and in the general case (*d*)
$$n\left(\frac{\lambda_n}{2}\right) = l.$$

This gives $\lambda_n = 2l/n$ instead of Equation (4) and leads to
$$f_n = \frac{n}{2l}\sqrt{\frac{T}{m_0}} \quad (n = 1, 2, 3, \text{etc.}) \quad . \quad (6)$$
instead of Equation (5).

Equation (6) indicates that a string of fixed length and tension vibrating in its fundamental mode (i.e. $n = 1$) will produce a note of a given frequency, equal to $\frac{1}{2l}\sqrt{\frac{T}{m_0}}$, and known as its fundamental frequency. But as it is made to vibrate in subsequent modes, i.e. $n = 2$, 3, etc., it will emit notes of double, treble, etc., the fundamental frequency; these are called *Harmonics*.

The note whose frequency is twice the fundamental is the second harmonic and so on. The second harmonic would be produced by a string vibrating as in Fig. 11.15 (*b*), which is said to be vibrating in its second harmonic mode. The string when vibrating in any one of these harmonic modes is in a resonant condition. If the passage of a travelling wave up and down the string is traced out, it will be found that in the second harmonic mode a wave initiated in the string by a source, after reflection begins its second run down the string in step with the next wave but one coming in from the source. It is then apparent that a vibrating system such as a stretched string can resonate at its fundamental frequency and at all the harmonics of the fundamental.

It would now be possible to examine theoretically the resonant modes of a string under tension but free at both ends, or fixed at one end and free at the other. Since, however, these conditions cannot be realised experimentally, the discussion will be reserved until later when dealing with vibrating air columns, which can be made to satisfy these conditions.

An experiment devised by Melde offers an experimental method of verifying Equation (6). A light cord is attached to one arm of an electrically maintained tuning-fork as in Fig. 11.16; the cord passes

over a pulley at the far end and is tensioned by weights placed in a pan.

If the fork is set in motion, a vibration of fixed frequency f is im-

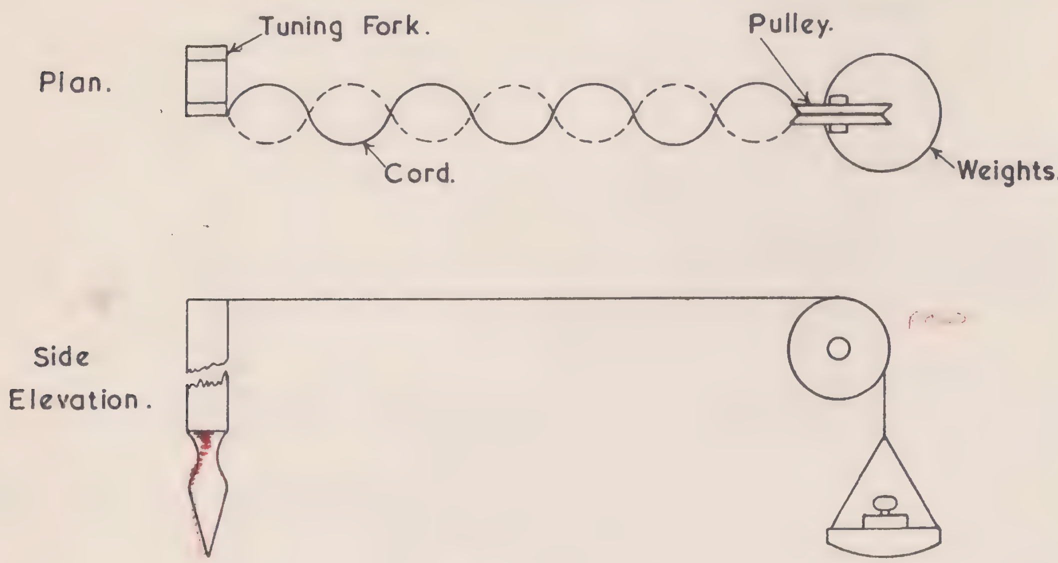


Fig. 11.16

pressed on the cord. The length of the cord and the tension in it are now adjusted until a standing wave of large amplitude is set up. This will occur only when the length of the cord is an integral number of half-wavelengths of the travelling wave induced in the cord by the tuning-fork. The tension in the cord adjusts the wavelength and the variation in length of the cord produces the necessary 'fit' of the waves on the string.

In this case, Equation (6) becomes:

$$f = \frac{n}{2l}\sqrt{\frac{T}{m_0}}$$

where n is the number of half-wavelength loops on the vibrating string; this gives:

$$f^2 = \frac{n^2}{4l^2} \cdot \frac{T}{m_0}$$

$$\text{or } \frac{n^2 T}{l^2} = 4m_0 f^2.$$

Now $4m_0f^2$ is a constant, thus any setting of tension and length which gives a standing wave of n loops should satisfy

$$n^2T/l^2 = \text{constant}.$$

Alternatively:

$$n^2T = (4m_0f^2)l^2 \quad . \quad . \quad . \quad . \quad (7)$$

thus a graph of n^2T against l^2 should give a straight line passing through the origin and of slope $4m_0f^2$ if Equation (6) is true.

In practice, the experiment does not permit of high accuracy.

11.8 The Sonometer

The most usual design of sonometer consists of a strong baseboard fitted with fixed and movable bridges as shown in Fig. 11.17. Two lengths of piano wire are stretched over the bridges, one usually being tensioned by a spindle turned by a key and the other by a weight hanging on the wire after it passes over a pulley. Either wire can be tuned to give the same note as a source of sound by varying the tension in the wire and the vibrating length of wire between the two bridges.

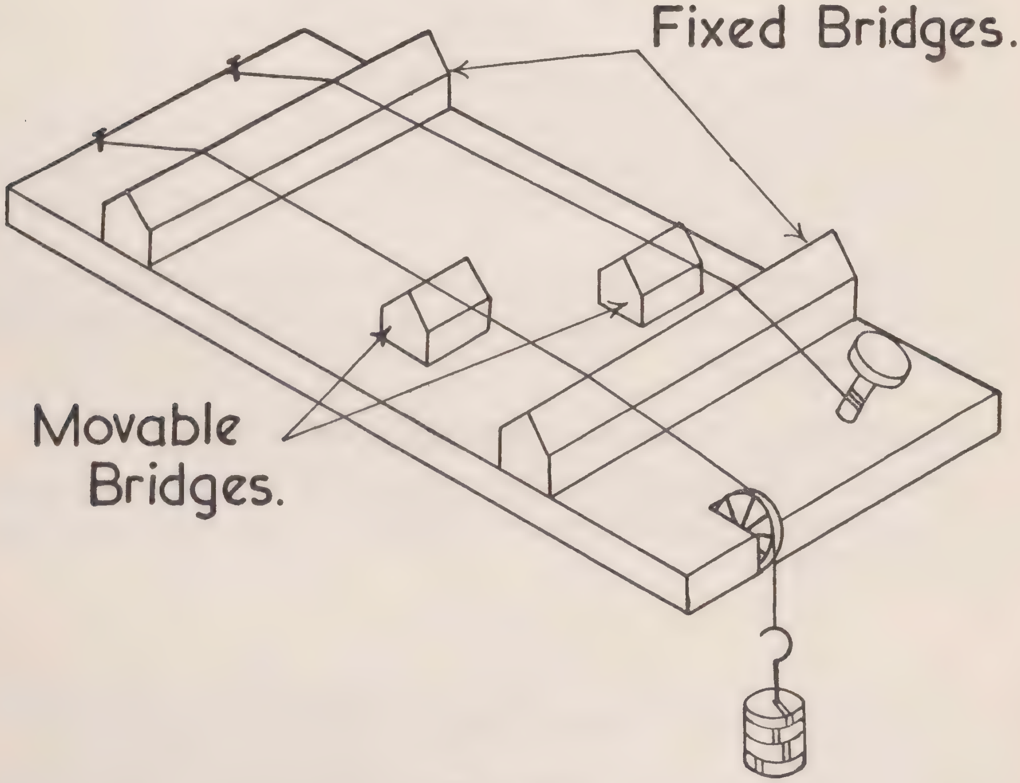


Fig. 11.17

Tuning presents a difficulty to some people without a musical ear; there are, however, experimental methods for bringing the wire into tune which do not rely on the ear of the experimenter. No one should attempt to avoid working with a sonometer just because he 'has no ear for music'.

The string tensioned by a key does not appear in all sonometers, but it is very useful when provided, for the string can be tuned to any note, which is thereupon 'stored up' in the string for reference when needed.

The baseboard, besides providing a rigid support for the wires, acts as a sounding-board. A vibrating wire cannot impart a very large vibration to the surrounding air and hence does not produce a very loud sound. If, however, the wire can set the baseboard in vibration through the bridges, the large board can hand on an increased vibration to the surrounding air and hence magnifies considerably the sound wave produced by the wire.

(a) To Measure a Frequency with a Sonometer

Weights are added or removed from the wire until it is roughly in tune with a source of sound, using about three-quarters of the full length of the wire as the vibrating portion. The movable bridge is then adjusted to bring the wire accurately into tune. If the mass hanging on the wire is M, the tension in the wire is Mg, and if the length of wire needed for resonance is l, then the frequency is given by Equation (6) as:

$$f=\frac{1}{2l}\sqrt{\frac{Mg}{m_0}} \quad . \quad . \quad . \quad . \quad (8)$$

where m_0 is the mass per unit length of the wire.

The method is not of high accuracy, for the tension may be influenced by friction in the pulley and at any point where the wire crosses a bridge, also it is difficult to determine the position giving the best tuning of the wire.

For details of experiments using a sonometer, the student is referred to *Experimental Physics*, by Daish and Fender, published by English Universities Press, Ltd.

11.9 Longitudinal Waves in a Stretched Wire

It was seen in the previous chapter (page 290) that it is possible to excite longitudinal vibrations in an elastic rod. The velocity of such waves is given by:

$$v_L=\sqrt{\frac{Y}{\varrho}} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where Y is the value of Young's Modulus and ϱ the density of the material of the rod.

Longitudinal waves can be excited in a stretched wire by gripping it in a cloth impregnated with powdered resin, and drawing the cloth along the wire—usually a high-pitched shriek will be heard; this is the result of a longitudinal standing wave in the wire.

Both ends of the wire must be very firmly clamped if they are to be nodes—a longitudinal wave passes unaffected over a bridge or a pulley. Even weights hung at the end of a wire cannot be considered as a rigid termination, for they can often be heard chattering when the wire is in vibration. This shows that the weights are performing an oscillation and are not at a node. If, however, the ends are firmly clamped and the wire is vibrating in its fundamental mode, i.e. a node at each end and just one half wavelength loop of standing wave on the wire, then the wavelength of longitudinal waves is given by

$$\frac{\lambda_L}{2}=l_L \quad . \quad . \quad . \quad . \quad . \quad (10)$$

if l_L is the total length of the wire.

But
$$v_L = f\lambda_L$$
$$= 2fl_L \qquad (11)$$

and thus, from Equation (9)

$$2fl_L = \sqrt{\frac{Y}{\varrho}} \qquad (12)$$

Notice that for most metals Y is about 20×10^{11} dyne.cm^{-2} and ϱ about 8 gm.cm^{-3}; this means that the frequency of longitudinal vibrations will be fairly high. For example a metre wire would give a frequency of

$$f = \frac{1}{200}\sqrt{\frac{20 \times 10^{11}}{8}}$$
$$= 2500 \text{ cps},$$

which is roughly the same pitch as the highest note on a piano.

It is possible to excite both longitudinal and transverse waves in a wire at the same time. If a pair of bridges is used to isolate a small portion in the middle of a long wire clamped at both ends, they will have no effect on longitudinal vibrations, but transverse vibrations can be produced between the bridges by plucking the wire. These bridges are moved until the frequency of transverse vibrations is the same as that for longitudinal vibrations.

The wavelength of transverse vibrations is given by $\lambda_T/2 = l_T$, where l_T is the distance between the two bridges.

But $v_T = f\lambda_T$
$$= 2fl_T,$$

also
$$v_T = \sqrt{\frac{T}{m_0}} \qquad (13)$$

where T is the tension in the wire and m_0 is the mass per unit length of the wire.

Thus
$$2fl_T = \sqrt{\frac{T}{m_0}} \qquad (14)$$

Dividing corresponding sides of Equation (14) by Equation (12) gives:

$$\frac{l_T}{l_L} = \sqrt{\frac{T\varrho}{Ym_0}} \qquad (15)$$

Further, if the wire is of cross-sectional area A, and a length L has a mass M, then:

$$\varrho = M/LA,$$
and $m_0 = M/L.$
Thus $\varrho/m_0 = 1/A.$

Substituting this in Equation (15) gives:

$$\frac{l_T}{l_L} = \sqrt{\frac{T}{YA}}$$

$$\text{or } A\left(\frac{l_T}{l_L}\right)^2 = \frac{T}{Y} \qquad . \quad . \quad . \quad . \quad . \quad (16)$$

Thus, since A, l_T and l_L can all be measured, either the tension in the wire or Young's Modulus for the material of the wire can be found if the other is known. The tension in the wire is usually a rather indeterminate quantity, since the wire must be strained between a pair of clamps, thus it is best to assume Young's Modulus and use this experiment to find the tension in the wire; alternatively, the tension can be found by a subsidiary experiment and so Young's Modulus can be calculated.

11.10 Vibration of Air Columns

A vibrating column of air enclosed in a tube is the basis of many musical instruments. In the subsequent paragraphs, therefore, the formation of standing waves under these conditions will be discussed and the possible modes of vibration explained; this will permit of a more ready understanding of the functioning of musical instruments when they are treated in the next chapter.

(a) Particle Displacement and Pressure Waves

If a standing wave is set up in a column of air it exists both as a standing wave of particle displacement and as a standing wave of pressure. Like travelling waves, the pressure and displacement waves have their maxima a quarter of a wavelength apart. If the displacement standing wave in an air column is shown by the solid curve of Fig. 11.18, then, remembering that the vibration is longitudinal, the particle displacement will be shown by the arrows. At points such as

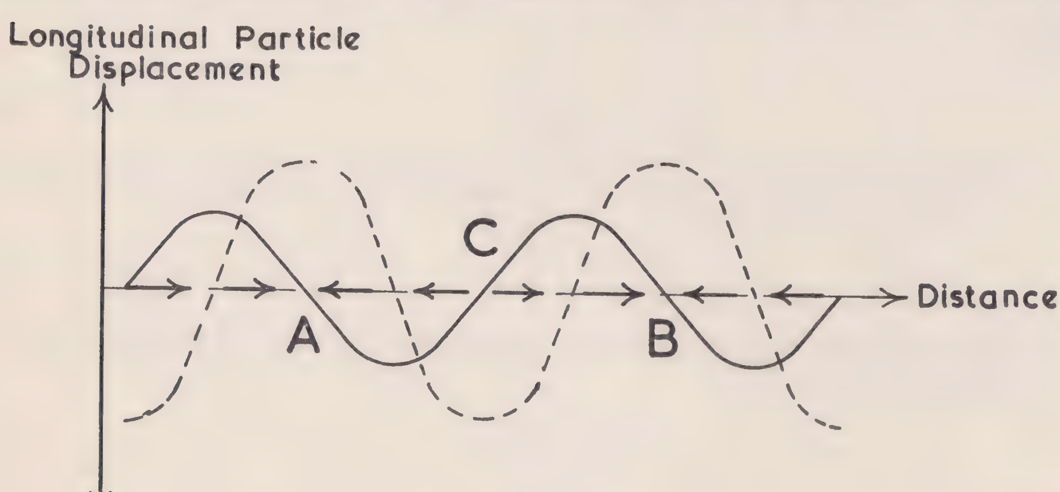


Fig. 11.18

A and B, the particles of the gas will be crowded together and so a pressure maximum will occur, while at C, particles are removed in both directions and consequently the pressure is lowered. Either pressure or displacement waves can be used to fix the position of the standing waves on a physical system, but we must always be careful to distinguish between the two sorts of waves.

(b) Tube Closed at One End and Open at the Other

The air in a tube can be set into vibration in many ways, most of which are described in the next chapter under the heading of musical instruments; for the theoretical discussion here we shall generally imagine that a tuning-fork is sounded and held near an open end of the tube as in Figs. 11.19 and 11.20, which show two typical laboratory methods of experimenting on vibrating air columns.

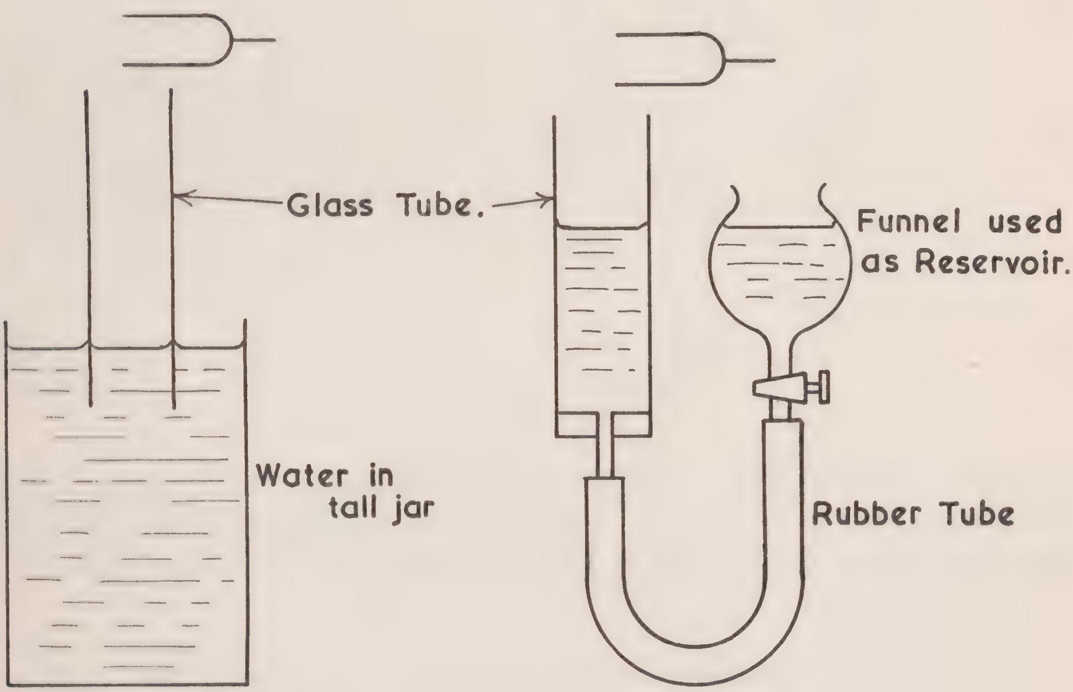


Fig. 11.19　Fig. 11.20

In Fig. 11.19 the glass tube is held in a gas jar filled with water, the water forms a closed end to the tube and so the length of the tube can be varied by raising or lowering it in the water.

The apparatus shown in Fig. 11.20 allows the level of the water in the glass tube to be altered by raising or lowering the height of the reservoir. In either case the lower end of the glass tube is closed by the water surface.

Now the water surface is not capable of vibration like the particles of air in the tube, thus the surface acts as a rigid boundary and prevents any particle displacement taking place. Any standing wave formed in the tube must therefore be one which places a displacement

node at the bottom or closed end. The upper end of the tube is open to the atmosphere, which will absorb any pressure changes occurring there; this point therefore must be a pressure node and hence a displacement antinode. Some possible modes of vibration of the air in the

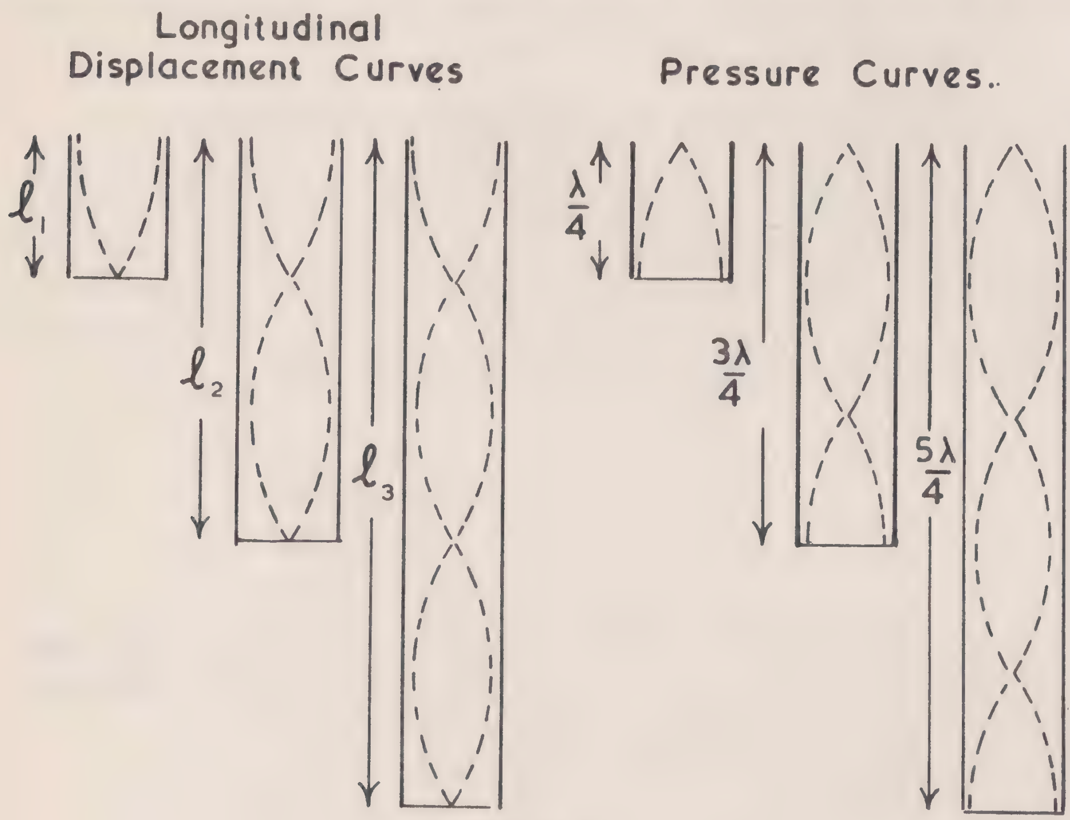


Fig. 11.21

tube are shown in Fig. 11.21. From this diagram it will be seen that the tube responds to a note which has a wavelength λ in air only when the length of the tube is $\lambda/4$, $3\lambda/4$, $5\lambda/4$, etc.

(c) End Effects

The open end of a tube is a true pressure node only if the pressure fluctuations emerging from the tube are absorbed by the air *without movement*. This of course is quite impossible, the air for a short distance

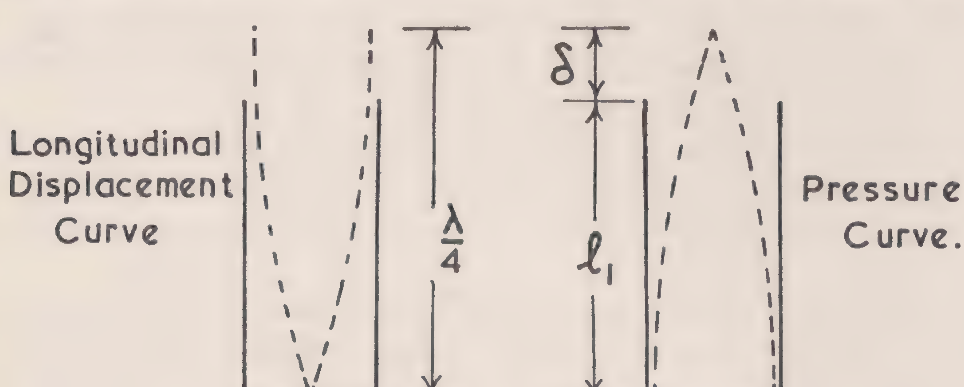


Fig. 11.22

outside the tube also vibrates, and thus the displacement antinode occurs at a point just outside the tube as in Fig. 11.22 and not as in Fig. 11.21. If this antinode is at a distance δ from the end of the tube, then from the diagram,

$$l_1 + \delta = \lambda/4.$$

Similarly, in the other cases shown in Fig. 11.21 the antinode at the open end will occur at a distance δ outside the tube, and thus

$$l_2 + \delta = 3\lambda/4,$$

$$l_3 + \delta = 5\lambda/4.$$

If, however, we subtract any of these equations from the next, we have

$$l_2 - l_1 = \lambda/2,$$

$$l_3 - l_2 = \lambda/2.$$

Thus the end effect is eliminated from the equation, and the difference between any two consecutive resonant lengths of the tube is a true half-wavelength. This can be used as an experimental method to measure the velocity of sound in air.

We have $v = f\lambda$,

where f is the frequency of the travelling wave forming the standing wave, i.e. the frequency of the tuning-fork in this case.

Thus $$v = 2f(l_2 - l_1) \quad . \quad . \quad . \quad . \quad (17)$$

from which the velocity of sound can be calculated.

It will be noticed that if the tube is kept of constant length, but is energised with travelling waves of various wavelengths, then (neglect-

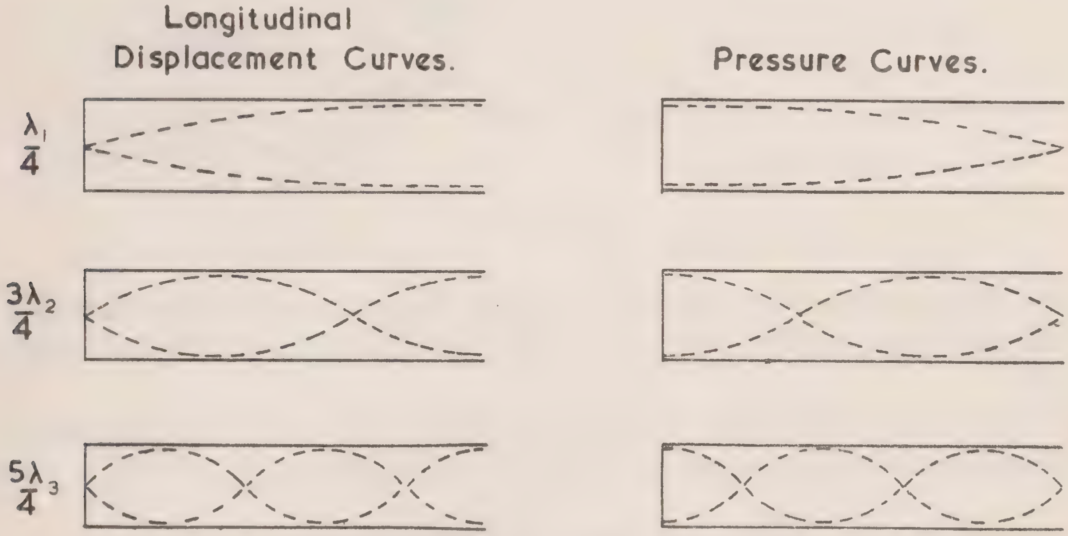


Fig. 11.23

ing the end effect) standing waves can be formed as shown in Fig. 11.23, i.e. the tube contains 1, 3, 5, etc., quarter-wavelengths of standing wave. Thus if the tube is of length l,

$$(2n - 1)\frac{\lambda_n}{4} = l, \text{ where } n = 1, 2, 3, \text{ etc.}$$

But the velocity of travelling waves in the tube is given by

$$v = f_n \lambda_n$$

Thus
$$\frac{(2n-1)}{4f_n} v = l$$

$$\text{or } f_n = (2n - 1)\, v/4l \quad (n = 1, 2, 3, \text{ etc.}) \quad . \quad (18)$$

The tube thus resonates to frequencies equal to $(v/4l)$, $3(v/4l)$, $5(v/4l)$, etc., i.e. to a fundamental and all the *odd* harmonics, but not the even harmonics.

These harmonics are sometimes called *Overtones*; the first overtone is the harmonic of lowest frequency produced by the vibrating system, and so on. In this case, the first overtone is the third harmonic, the second overtone is the fifth harmonic, etc.

(d) Tube Closed at Both Ends

The air column in a tube closed at both ends can be set into vibration if one of the end plates is a diaphragm of a telephone receiver (see Chapter 12). An alternating electric current fed into the energising coils of the receiver will vibrate the diaphragm and can be used to excite longitudinal vibrations in the air column; the arrangement is shown in Fig. 11.24.

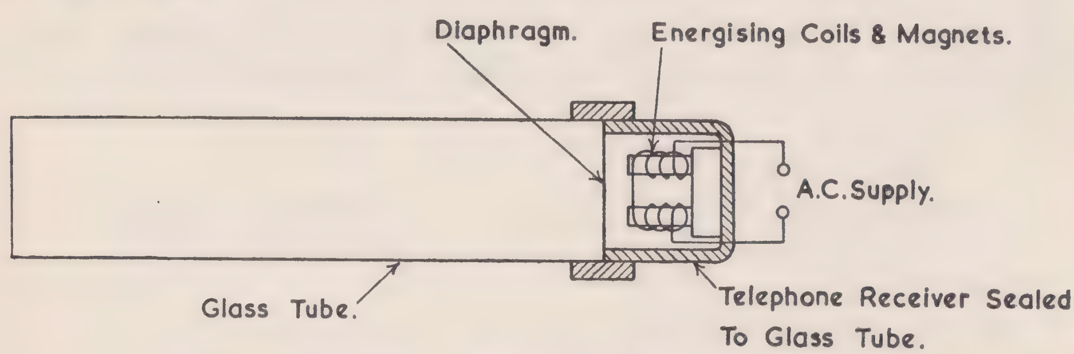


Fig. 11.24

Since both ends of the tube are closed, they must be displacement nodes and thus standing waves can 'fit' into the tube as shown in Fig. 11.25.

It will be seen that the tube length divides into 1, 2, 3, etc., half-wavelengths of standing wave; thus if l is the length of the tube,

$$\frac{n\lambda_n}{2} = l, \quad \text{where } n = 1, 2, 3, \text{ etc.}$$

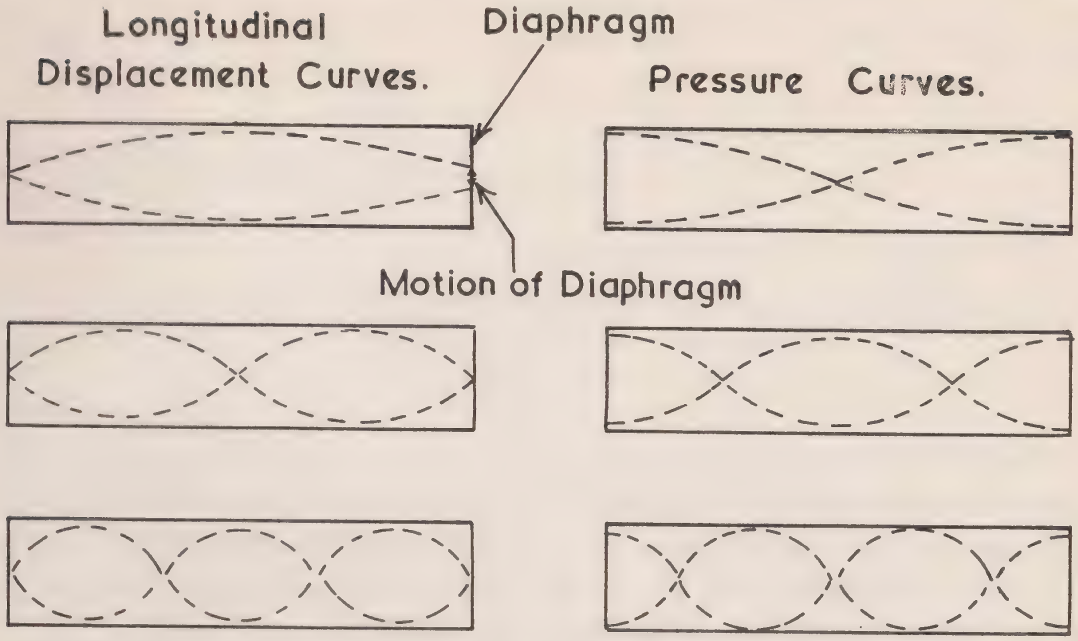


Fig. 11.25

But the velocity of the travelling wave in the tube is given by

$$v = f_n \lambda_n.$$

Thus
$$\frac{nv}{2f_n} = l$$

$$\text{or } f_n = nv/2l \quad (n = 1, 2, 3, \text{ etc.}) \qquad . \qquad . \quad (19)$$

The tube thus resonates to frequencies equal to

$$(v/2l),\ 2(v/2l),\ 3(v/2l), \text{ etc.},$$

i.e. to a fundamental and complete harmonic range.

The end of the tube closed by the diaphragm has been treated as a node, although the diaphragm is in vibration. This is permissible since the amplitude of the diaphragm motion is very small compared with the maximum particle displacement in the standing wave. It introduces an end correction, as illustrated in the top left-hand diagram, of Fig. 11.25; but the correction is usually very small and only of account in work of the highest precision.

(e) Kundt's Tube

The resonance of a tube closed at both ends forms the basis of yet another laboratory experiment. We have discussed above the various frequencies producing resonance in a tube of fixed length, but the experiment is better performed the other way round, i.e. by seeking the various lengths of tube which will resonate to a fixed frequency. The piece of apparatus used is known as Kundt's Tube and is shown in Fig. 11.26.

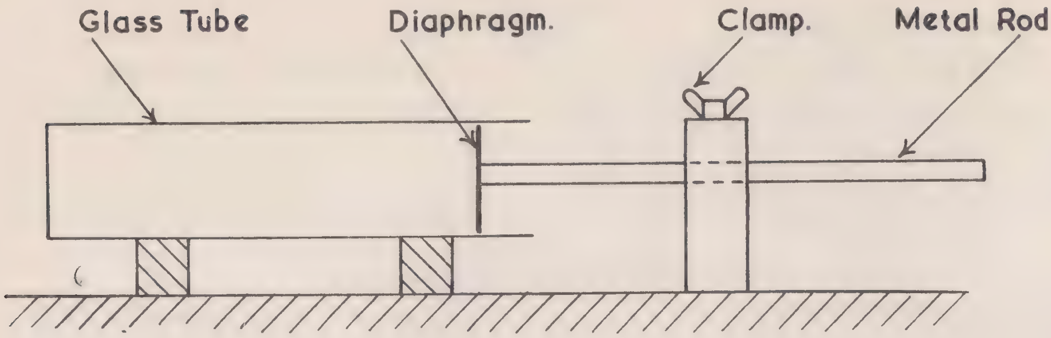


Fig. 11.26

A metal rod is rigidly clamped at its centre point and carries at one end a metal diaphragm which just fits inside a glass tube. The tube is closed at one end and is mounted on guide blocks so that it can be slid over the diaphragm, thus the length of air column between the closed end of the tube and the diaphragm can be varied.

The rod acts as a fixed source of frequency in the following way. If a cloth impregnated with resin is stroked with a firm grip along the outer half of the rod, longitudinal waves will be set up in the rod and a high-pitched note will be heard. The frequency of this note can be calculated as follows. The rod is clamped at its midpoint and free at the ends, hence the longitudinal standing wave set up on the rod will be

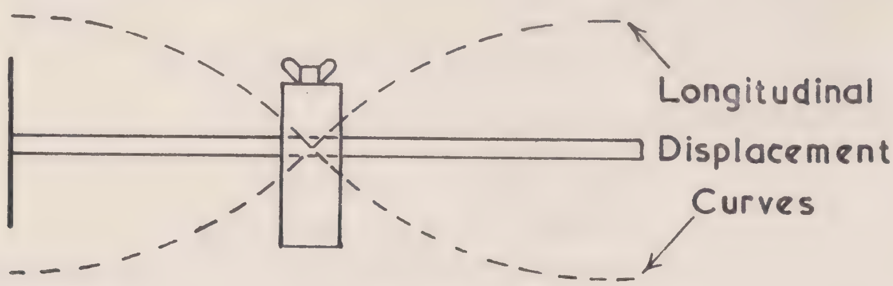


Fig. 11.27

as shown in Fig. 11.27, i.e. with a node at the fixed point and antinodes at the free ends. The rod thus carries half a wavelength of standing wave, or, if the rod is of length L, then

$$\lambda_R/2 = L,$$

where λ_R is the wavelength of the standing wave in the rod.

But
$$f = v_R/\lambda_R,$$

where v_R is the velocity of longitudinal waves in a rod, thus

$$f = v_R/2L.$$

But
$$v_R = \sqrt{Y/\varrho}$$

where Y is Young's Modulus for the material of the rod and ϱ is its density.

Thus $$f = \frac{1}{2L}\sqrt{Y/\varrho} \qquad (20)$$

Now the diaphragm at the end of the rod will excite longitudinal vibrations in the air column at the frequency given by Equation (20), and the wavelength λ_A of these waves in air will be given by

$$f = v_A/\lambda_A,$$

where v_A is the velocity of the waves in air. Thus, from Equation (20),

$$1/\lambda_A = \frac{1}{2Lv_A}\sqrt{\frac{Y}{\varrho}} \qquad (21)$$

Standing waves in the air in the tube form only when the tube is an integral number of half-wavelengths long, as was shown in Fig. 11.25. Thus whilst stroking the rod the glass tube is moved slowly to and fro until a standing wave of large amplitude builds up in the air column.

The standing wave is detected by placing a small amount of lycopodium powder in the tube. The particles of powder will take up some of the particle motion of the air, and the growth of the standing wave can be judged by the violence of the motion of the lycopodium particles.

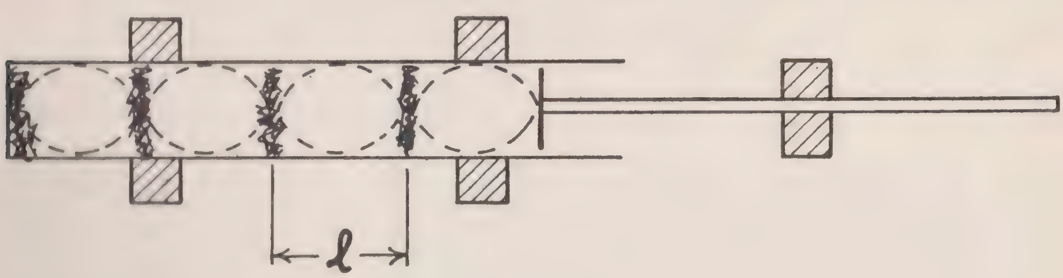


Fig. 11.28

The lycopodium particles also serve another useful purpose. Whilst being swept about by the vibrations of the air they first of all collect in groups of narrow ridges at the antinodal points (like ripples in sand, but on a very small scale); but if the excitation is continued long enough, the particles gradually move to the displacement nodes and come to rest in heaps there, as in Fig. 11.28. The distance l between any adjacent pair of nodal heaps (or groups of antinodal ridges) is equal to a half-wavelength of the travelling wave in air.

Hence $$l = \lambda_A/2,$$

and substituting this in Equation (21) gives:

$$v_A = \frac{l}{L}\sqrt{\frac{Y}{\varrho}} \qquad (22)$$

From this equation the velocity of sound in air can be calculated.

This experiment permits of many variations; for example the frequency of vibration of the rod may be measured with a sonometer, thus f is known and we have:

$$v_A = f/2l \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Alternatively, we may write

$$v_A = \sqrt{\gamma p/\sigma}$$

where γ is the ratio of the specific heats of air at constant pressure and constant volume, p is the pressure of the air and σ its density.

Thus
$$\sqrt{\frac{\gamma p}{\sigma}} = \frac{l}{L}\sqrt{\frac{Y}{\varrho}}$$

or
$$\gamma = \frac{\sigma Y}{p\varrho}\left(\frac{l}{L}\right)^2.$$

If we refer to a gram-molecule of gas, then

$$p = \frac{RT}{V} \text{ (see Chapter 7),}$$

and
$$\sigma = M/V,$$

where M is the molecular weight of the gas.

Thus
$$\sigma/p = M/RT,$$

and
$$\gamma = \frac{M}{RT} \cdot \frac{Y}{\varrho} \cdot \left(\frac{l}{L}\right)^2 . \quad . \quad . \quad . \quad . \quad (24)$$

Hence the ratio of the specific heats may be calculated since all the factors on the right-hand side can either be measured or are known constants. The rapid expansion and compression of a gas which occurs in a sound wave probably gives a nearer approach to adiabatic conditions than can be produced by any other method; thus a fairly accurate value for γ can be found.

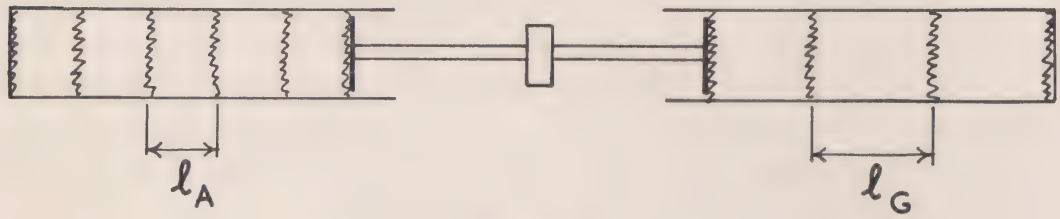


Fig. 11.29

A further variation consists of fitting each end of the rod with a diaphragm, both being fitted into glass tubes as in Fig. 11.29. One tube is filled with air and the second with another gas. The rod is stroked and travelling waves of the same frequency are excited in both tubes; the

tubes are then adjusted until standing waves occur, and the length of a half-wavelength measured in each case. Then, from Equation (23),

$$v_A = f/2l_A$$
$$\text{and } v_G = f/2l_G,$$

where v_G is the velocity of sound in the gas.

$$\text{Thus } \frac{v_A}{v_G} = \frac{l_G}{l_A}$$
$$\text{or } v_G = v_A l_A / v_G \qquad . \quad . \quad . \quad (25)$$

and hence the velocity of sound in any gas can be compared with that in air.

(f) Tube Open at Both Ends

This represents a case which is of importance in many musical instruments (see Chapter 12). If both ends of the pipe are open, then standing waves will be formed as shown in Fig. 11.30, i.e. with a pressure node at each end.

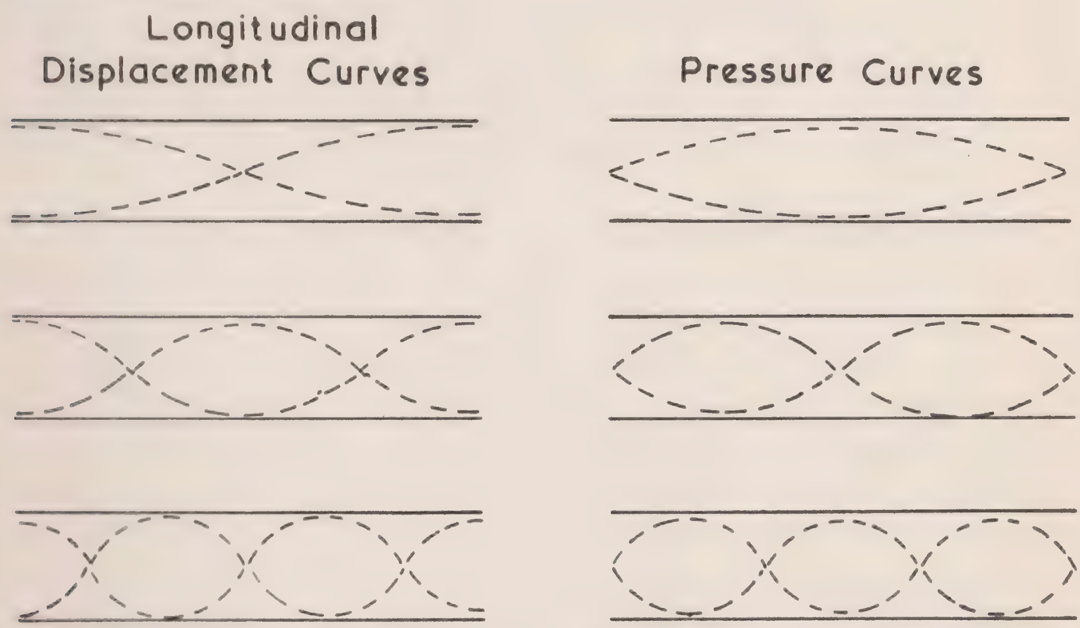


Fig. 11.30

It will be seen that a tube of length l contains 1, 2, 3, etc., half-wavelengths of standing waves; thus if λ_n is the wavelength of the travelling wave which forms n half-wavelength loops of standing wave in the tube, then:

$$n\lambda_n/2 = l, \quad \text{where } n = 1, 2, 3, \text{ etc.}$$
$$\text{or } \lambda_n = 2l/n.$$

But $v = f_n\lambda_n$, where v is the velocity of sound in the air in the pipe. Thus the pipe has a fundamental mode of frequency $f = v/2l$ and also resonates to frequencies $2(v/2l)$, $3(v/2l)$, etc., i.e. a full harmonic range.

Example. *The air column in a tube 36 cm long and open at both ends, and the column in a tube 28 cm long and closed at one end respond to the same note as their first overtone. The tubes are both cylindrical and of the same diameter. Find the frequency of the note and the end correction for the pipes. (The velocity of sound in the air in the tubes is 341 metres per second.)*

Assuming that the correction is the same for the open ends in each case, the modes of vibration of the two pipes are as shown in the diagrams.

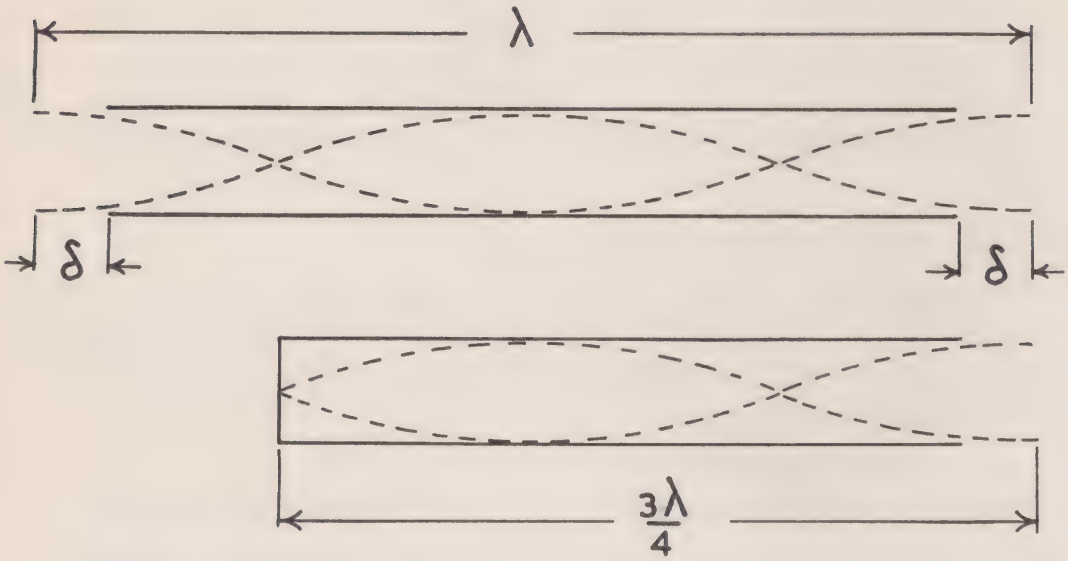


Fig. 11.31

If the wavelength of the sound is λ and the end correction is δ, then:

$$\lambda - 2\delta = 36 \text{ cm},$$

$$3\lambda/4 - \delta = 28 \text{ cm}.$$

$$\text{Whence } \lambda = 40 \text{ cm}.$$

$$\text{and } \delta = 2 \text{ cm}.$$

The frequency of the note is thus given by

$$f = \frac{341 \times 10^2}{40}$$

$$= 852{\cdot}5 \text{ cps}.$$

11.11 Helmholtz Resonators

The air contained in a bottle is capable of vibrating at a certain frequency, as anyone who has blown across the neck of a bottle will know. This is not quite the same phenomenon as the resonance which occurs in the air column in a pipe, but is due to the mass of air in the bottle acting rather as a spring and causing the small plug of air in the neck of the bottle to vibrate inwards and outwards.

An approximate theory can be developed as follows. A bottle has a neck of cross-sectional area A and length l, the volume of the bottle is V as shown in Fig. 11.32.

Consider the plug of air in the neck moving inwards by a distance x, then the volume of air δV compressed into the bottle is equal to Ax;

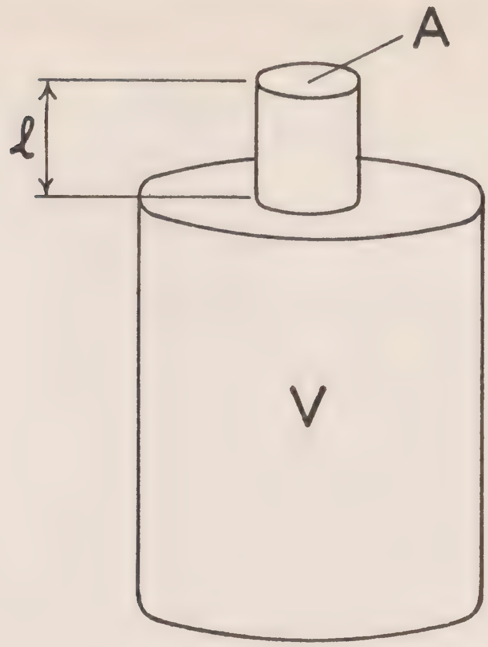


Fig. 11.32

this will cause an increase in pressure δp. If the change takes place very rapidly, adiabatic conditions will prevail and the gas will obey the law:

$$pV^{\gamma} = k$$

where k is some constant and γ is the ratio of the specific heats of the gas.

$$\text{Thus } p = kV^{\gamma}$$

Differentiating with respect to V gives

$$\frac{dp}{dV} = -\gamma k V^{(-\gamma-1)}$$

$$\text{But } kV^{(-\gamma-1)} = p/V,$$

$$\text{thus } dp/dV = -\gamma p/V.$$

Writing this in terms of small changes gives:

$$\delta p = -\gamma p \delta V/V.$$

Substituting the value of δV found above gives:

$$\delta p = -\gamma p A x/V.$$

The restoring force exerted by the compressed gas on the plug of air in the neck is $A\delta p$, thus:

$$\text{restoring force} = -\gamma p A^2 x/V.$$

Now this restoring force is proportional to the displacement x, thus the motion of the plug of air will be simple harmonic, with a period given by:

$$T = 2\pi \sqrt{\frac{\text{Mass}}{\text{Restoring force at unit displacement}}}.$$

If the density of the air is ϱ, then the vibrating mass in the neck is $Al\varrho$, thus

$$T = 2\pi\sqrt{\frac{Al\varrho}{\gamma p A^2/V}},$$
$$= 2\pi\sqrt{\frac{\varrho}{\gamma p}\,.\,V\,.\,\frac{l}{A}}\,.$$

But the velocity of sound in air is given by

$$c = \sqrt{\frac{\gamma p}{\varrho}},$$

$$\text{thus } T = \frac{2\pi}{c}\sqrt{\frac{Vl}{A}} \quad . \quad . \quad . \quad . \quad (26)$$

And the resonant frequency $f = 1/T$ is given by

$$f = \frac{c}{2\pi}\sqrt{\frac{A/l}{V}} \quad . \quad . \quad . \quad . \quad (27)$$

The factor A/l is called the 'conductivity' of the neck of the resonator.

It is obvious that the theory given above is only an approximation; for example, it has been assumed that only the mass of air in the neck of the bottle is vibrating and further that it vibrates as a solid incompressible plug, neither of which is true. Nevertheless, the complete theory of the resonator leads to an expression of the same form as Equation (27) but with a function more complex than A/l for the conductivity of the neck. If we write S for this conductivity, then

$$f = \frac{c}{2\pi}\sqrt{\frac{S}{V}}$$
$$\text{or } f^2V = c^2S/4\pi^2.$$

But for any given bottle, $c^2S/4\pi^2$ is a constant, thus

$$f^2V = \text{constant} \quad . \quad . \quad . \quad . \quad (28)$$

This expression can be demonstrated experimentally to be true.

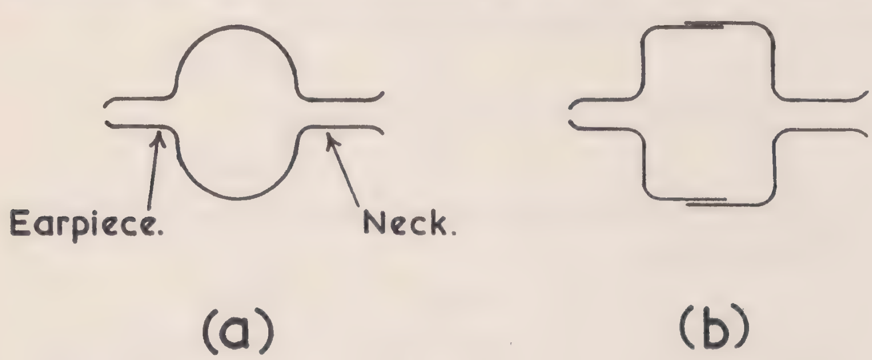


Fig. 11.33

Helmholtz made resonators in this fashion to assist him in his researches on harmonics and overtones. He used small brass vessels, as shown in Fig. 11.33, the size of the cavity and neck being so adjusted

that the vessel resonated to the desired note. The resonator was also provided with an ear-piece which could be fitted into the ear, and so a resonance, even to a very faint note, could be heard. A whole series of resonators is of course needed to cover a musical scale, but if the volume of the resonator can be adjusted, as in the pattern shown in Fig. 11.33 (*b*), then it can be tuned to any note within quite a wide range.

11.12 Vibrations of Flat Plates

Transverse waves may be excited in a flat plate by clamping the plate at the centre and bowing it at the edge as in Fig. 11.34. These waves spread out in the plate, are reflected from the edges, and form a very complex pattern of standing waves. The theory of such standing

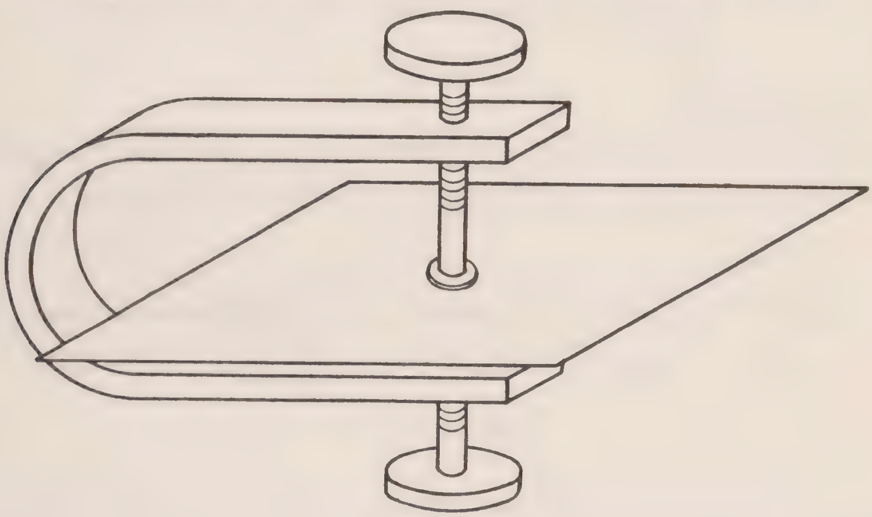


Fig. 11.34

waves is complicated, but they can be studied experimentally by scattering fine sand on the plate. When this is set in vibration, the sand moves to the nodes, forming a pattern which allows the nodes of vibration of the plate to be examined. Chladni carried out a series of experiments on vibrating plates, and some of the modes of vibration

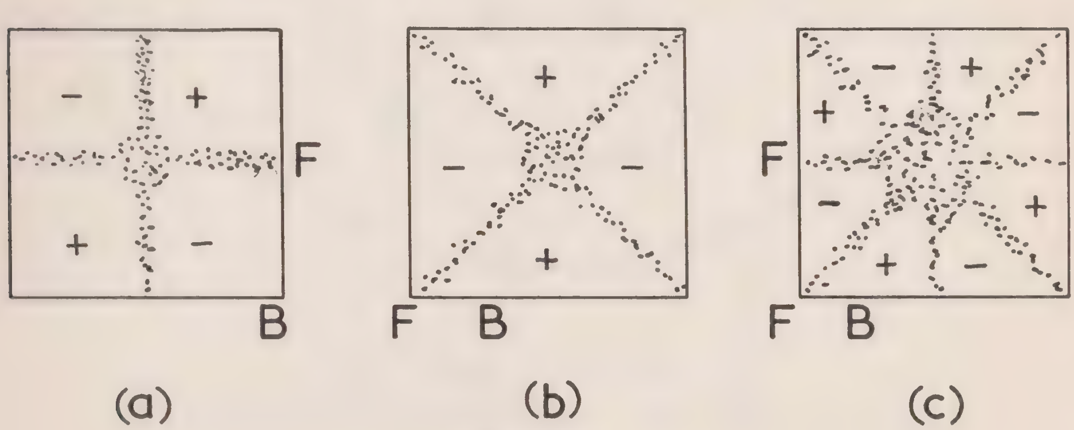


Fig. 11.35

which he discovered are shown in Fig. 11.35. In each diagram B indicates the point at which the plate is bowed, F the point at which it is touched with the finger so producing a node, whilst the signs + and − indicate portions of the plate which at any instant are moving in opposite directions.

The direction of motion of parts of the plate was discovered by Lissajous. This can be demonstrated experimentally by sounding the plate in the mode shown in Fig. 11.34 (a) and then covering one quarter of the plate with a piece of card, whereupon the sound heard from the plate is increased. The unobstructed plate would emit rarefaction waves from two segments and compression waves from the others; these largely cancel each other out. But if one segment is obscured the cancellation will be reduced and the volume of sound increased.

11.13 Tuning-forks

The theory of the tuning-fork may be developed from experiments made by Chladni on the transverse vibrations of a bent bar.

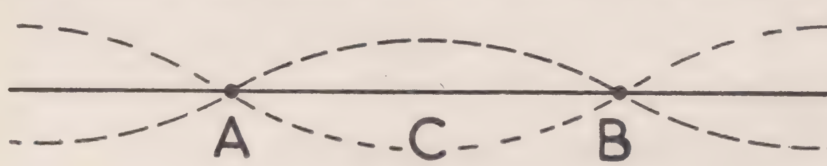


Fig. 11.36

A long straight bar with both ends free can vibrate as shown in Fig. 11.36. Such a vibration could be excited by clamping the bar at A and B, which are nodes, and striking it at the antinode C.

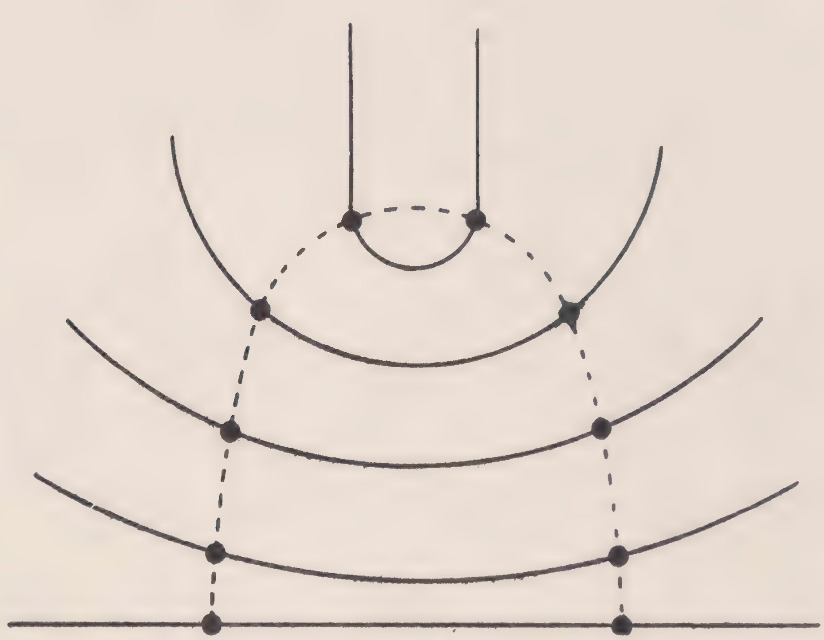
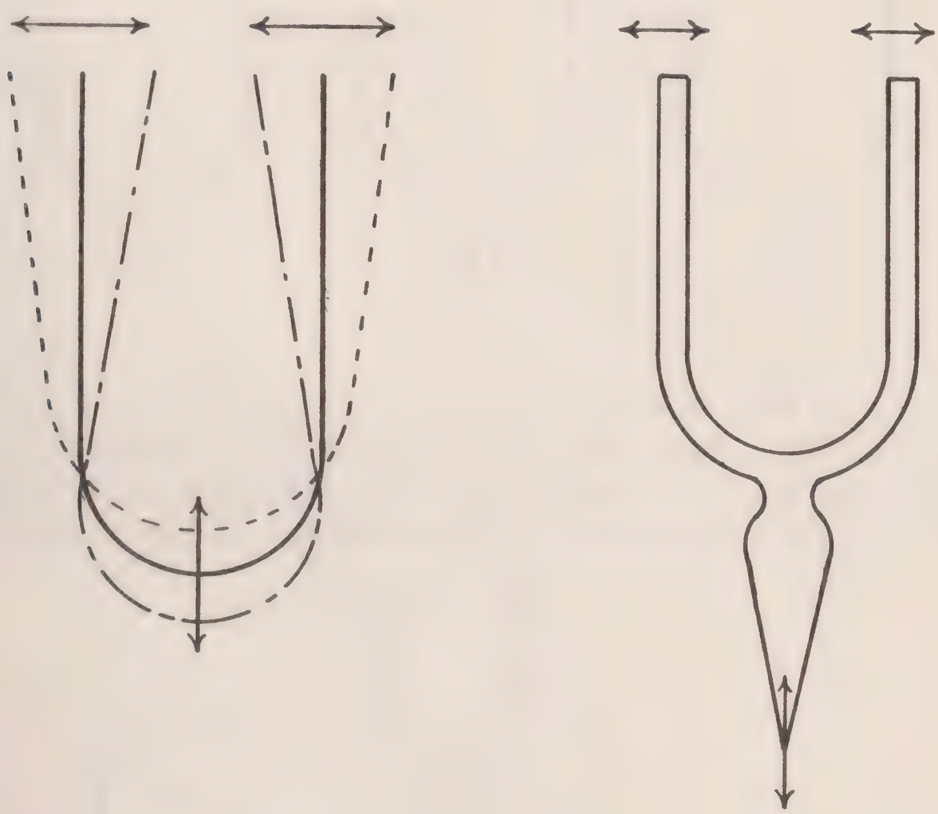


Fig. 11.37

Chladni traced out the movement of the nodes as the bar is bent into a U-shape, with the result shown in Fig. 11.37. From the diagram it is seen that as the bar is bent, the nodes move closer and closer together until they reach the bottom of the straight arms where the bar becomes completely U-shaped; this, of course, is the shape adopted as a tuning-fork.



(a) (b)

Fig. 11.38

When a tuning-fork is vibrating, successive positions of the standing wave will be as shown in Fig. 11.38 (*a*), hence a side-to-side motion of the prongs will result in the up-and-down motion of a stem (shown in Fig. 11.38 (*b*)). If this stem is pressed against some large surface such as a table-top, the up-and-down motion is communicated to the surface. This in turn excites longitudinal waves in the air, and so a fairly loud sound can be heard from the fork.

A tuning-fork has modes of vibration other than that shown in Fig. 11.38 (*a*); the common ones are shown in Fig. 11.39. The tones produced, however, are not harmonics of the fundamental; the mode shown in Fig. 11.39 (*b*) produces a note between 2 and 3 octaves above the fundamental while the mode shown in Fig. 11.39 (*c*) produces a note just over 4 octaves higher than the fundamental.

Some tuning-forks are mounted on a resonator box (see Fig. 11.40). This is an open-ended box of such a volume that the air resonates to the

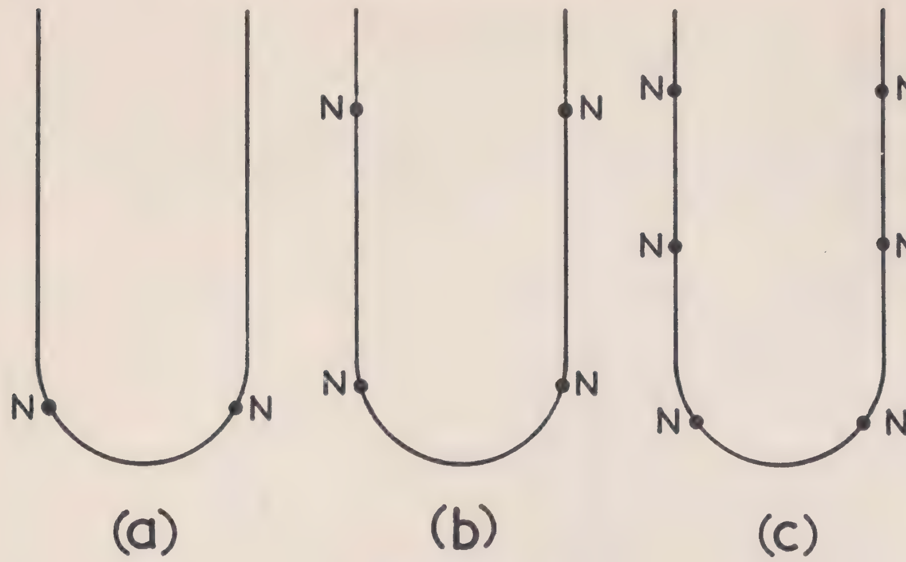


Fig. 11.39

fundamental frequency of the fork. The fork sets one side of the box in vibration, this in turn causes the air in the box to resonate, and the vibration of the air in the mouth of the box emits large travelling waves into the atmosphere—much larger than could be caused by the

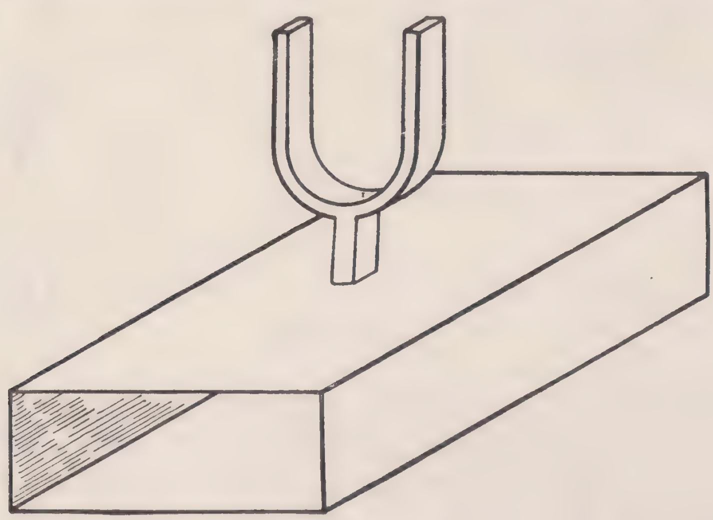


Fig. 11.40

small vibrating surfaces of the fork. Moreover, the box resonates only to the fundamental of the fork and not to its overtones, for they are not harmonics; thus the sound emerging from the resonator box is very nearly a pure tone.

A tuning-fork provides a very stable frequency standard; even the handling received in an elementary laboratory does not alter its frequency appreciably. Temperature changes cause a slight fluctuation in frequency, but the frequency of a steel fork decreases by only 0·01 per cent. for every degree Centigrade rise in temperature, thus this correction is needed only for work of the highest accuracy.

EXERCISES 11

1. Interpret the expression $y = a \sin\frac{2\pi}{\lambda}(vt - x)$ as applied to the propagation of sound waves. Discuss the nature and characteristics of the resulting motion due to the superposition of two identical wave trains moving in opposite directions. Give a short account of any acoustic measurement utilising such a system.

   A train of plane sound waves traverses a medium and the individual particles execute a periodic motion such that their displacement (in cm.) is given by $y = 5 \times 10^{-6} \sin(800\pi t + \theta)$.

   (i) What is the amplitude of the particle motion?
   (ii) Calculate the wavelength of the waves.
   (iii) Calculate the phase difference in degrees between two particles 17 cm. apart, at any *given instant*.

   (Velocity of sound in medium $= 3{\cdot}4 \times 10^4$ cm./sec.)

   (London Univ. G.C.E. Schol. level.)

2. You are provided with two tuning-forks of unknown, but very close, frequencies. If the facilities of an ordinary laboratory are available, describe how you would determine (*a*) the difference in frequency between the forks, (*b*) which fork has the higher frequency, and (*c*) the absolute frequency of either fork.

   Two organ pipes, open at both ends, 90 cm. and 91 cm. long, when sounding their fundamental notes are found to give 41 beats in 20 sec. Ignoring end corrections, deduce the frequencies of the two notes and the velocity of sound in air.

   (London Univ. G.C.E. Advanced level.)

3. Derive an expression for the velocity of propagation of a transverse wave along a string.

   A steel wire of density 7·9 gm.cm.$^{-3}$ and diameter 0·45 mm. is clamped tightly between rigid supports 20 cm. apart under a tension of 10 Kgm. Find the frequency of the lowest note which it emits when plucked. (Manchester Univ. Schol.)

4. Write down expressions for the velocity of propagation along a stretched wire of (*a*) transverse waves, (*b*) longitudinal waves. Deduce *one* of the expressions.

   Find the ratio of the fundamental frequencies of transverse and longitudinal vibrations for a steel wire, diameter 1 mm., mounted on a sonometer and stretched by a force of 10 kilograms weight. (Young's modulus for steel $= 20 \times 10^{11}$ dynes per sq.cm.)

   Describe and explain a method of finding the velocity of longitudinal waves in a long steel rod.

   (Northern Univ. H.S.C. Schol. level.)

5. Describe an experiment to determine how the length, l, of a stretched wire must be varied to keep its frequency of transverse vibration constant when the tension, T, is altered. Give an example of the kind of graph obtained if $\log l$ is plotted as ordinate against $\log T$ as abscissa.

What is the intercept on the log T axis when log $l = 0$ if the frequency is 200 vibrations per sec. and the linear density of the wire is 0·0169 gm. per cm.? (London Univ. Inter. B.Sc.)

6. Describe with diagrams the nature of the particle motion of a stretched string when excited into (*a*) transverse and (*b*) longitudinal vibrations. What are the physical factors controlling the velocity of propagation in each case?

Describe briefly how you would use a wire sonometer to compare the frequencies of two tuning-forks.

The velocity of transverse waves along a steel piano wire under a certain tension is 2×10^4 cm./sec. and its fundamental frequency is 200 c/s. Calculate (i) the length of the wire and (ii) the number of the overtone corresponding to a frequency of 1,200 c/s.

(London Univ. G.C.E. Advanced level.)

7. Describe an experiment to investigate the way in which the fundamental frequency of transverse vibration of a stretched string depends on the length of the string and the tension applied to it.

Two strings of the same material, cross-sectional area, and length, are stretched on a sonometer board. The tensions are in the ratio 1·8 to 1, and when the strings are sounded together 4 beats per second occur between the third harmonic of the tauter string and the fourth harmonic of the other. Find the fundamental frequency of each.

(Oxford G.C.E. Advanced level.)

8. Describe how you would measure the velocity of sound in air by a simple resonance tube method. Give the theory underlying your method and state the precautions you would take to obtain an accurate result.

A steel wire of diameter 0·05 cm. is fixed at one end and is stretched horizontally by a load of 10 kgm. hanging over a small smooth pulley at the other end of the wire. Find the frequency of the fundamental note emitted when the horizontal portion of the wire is set into transverse vibration, given that this portion of the wire extends by 0·125 cm. when it is loaded by a weight of 5 kgm. (The density of steel is 7·7 gm. per c.c. and Young's modulus is 2×10^{12} dynes per sq.cm.)

(Oxford G.C.E. Schol. level.)

9. Distinguish between progressive and stationary waves, and describe how you would use a tuning-fork or other device vibrating at constant frequency to maintain a stationary wave system in a stretched string.

Two strings of the same material are stretched side by side on a sonometer to the same tension, and they are adjusted to unison with a fork of frequency 328 per sec. Their lengths between the bridges are 75·0 and 90·0 cm. respectively. Find the ratio of the diameters of the two wires, and also the number of beats per second that will be heard if the length of the longer wire is increased to 90·3 cm.

(Oxford G.C.E. Advanced level.)

10. A sonometer wire is stretched between two bridges on a large wooden mount. When it is plucked sharply in the middle and released a musical note is heard. Describe carefully, and as fully as you can,

what is happening during the sounding of this note (*a*) to the wire; (*b*) to the mount; (*c*) to the air between the mount and the observer's ear.

The length of the wire is 80·0 cm., and its tension is so adjusted that it is in unison with a fork of frequency 256 $sec.^{-1}$ One bridge is accidentally displaced so that the separation becomes 80·4 cm. Calculate the frequency of the beats now obtained when the fork and the wire are sounded together, and find the percentage alteration in the tension which would restore the pitch of the note to the original value. (Oxford H.S.C.)

11. Describe a sonometer method of determining the frequency of the A.C. supply.

A lump of metal hangs freely in air from one end of a sonometer wire whose resonant length is 80 cm. when tuned to a certain fork. When the metal hangs wholly immersed in water the resonant length is 75 cm. Find the specific gravity of the metal. (London Univ. Inter. B.Sc.)

12. What is meant by a simple mode of vibration of a stretched string? How are the frequencies of the possible simple modes of vibration of a string related to one another?

Describe in detail an experiment to compare the frequency of a tuning-fork (approximately 400 $sec.^{-1}$) with that of the a.c. electric supply (50 $sec.^{-1}$). (London Univ. Inter. B.Sc.)

13. Describe in detail how to determine the frequency of a tuning-fork by the falling-plate method.

Describe how the frequency of an alternating current, such as that from the A.C. mains, may be determined with a sonometer (Northern Univ. H.S.C.)

14. Describe and explain, with the aid of a suitable diagram, the movements of the air in a tube closed at one end and sounding its fundamental note.

A series of observations of the minimum lengths L of the air column in a resonance tube closed at one end resounding to tuning-forks of various known frequencies n are plotted with the values of L as ordinates and those of $1/n$ as abscissæ. Sketch the graph you would expect to obtain and explain the significance of its slope and its intercept on the L axis ($1/n = 0$).

The first overtone of a pipe open at both ends is of the same frequency as the second overtone of a pipe closed at one end. Assuming the end corrections to be negligible, compare the lengths of the two pipes. (Northern Univ. G.C.E. Advanced level.)

15. Indicate, with the aid of diagrams, the first three (starting with the fundamental) modes of vibration of the air in an open organ pipe, and state the relation between their frequencies.

An open organ pipe is composed of two thin cylindrical tubes, one sliding closely over the other. The combined length of the tube is so adjusted that on blowing at a temperature of 15° C. it utters a tone making 4 beats per second with that of a standard fork of frequency

1000 cycles per second. On extending the length of the tube by 17·09 cm. the sound elicited on blowing gives once more 4 beats per second with the fork, the fork being at the higher frequency in each instance. Calculate from these data the velocity of sound in air at 0° C.

(London Univ. Inter. B.Sc.)

16. Distinguish between progressive and stationary waves, and describe how you would produce a stationary wave system in an air-filled tube. Draw a diagram indicating the distribution of the nodes and antinodes for both particle displacement and pressure variation along the tube.

A wind instrument of the 'closed pipe' type is blown with air at 30° C. Find the frequencies of the fundamental and the first overtone when the effective length of the air column is 40 cm. (The velocity of sound in air at 0° C. is $3{\cdot}31 \times 10^4$ cm. per sec.)

(Oxford G.C.E. Advanced level.)

17. Describe the possible modes of vibration of the air in a pipe open at both ends. How is the pitch of the note given by such a pipe affected (*a*) by the diameter of the pipe, (*b*) by atmospheric pressure, (*c*) by temperature?

The second overtone of a pipe open at both ends has the same frequency as the first overtone of a pipe of the same diameter closed at one end. Find the ratio of the lengths of the two pipes.

(London Univ. Inter. B.Sc.)

18. Explain the meaning of *resonance*.

How may the frequencies of two tuning-forks be compared by means of a resonance tube?

When a tuning-fork is held over a long tube containing water, resonance occurs when the water-level is 13·8 cm. from the open end of the tube and again at 43·8 cm. Find the frequency of the fork and the end correction of the tube, the velocity of sound being 340 metres per sec. at the temperature of the experiment.

(London Univ. Inter. B.Sc.)

19. A uniform brass rod, 80 cm. long, clamped at the centre, is used to produce dust figures in air at 20°C., the distance apart of successive heaps being 8·4 cm. Using the following data, calculate the ratio of the principal specific heats of air: Young's modulus for brass = $9{\cdot}0 \times 10^{11}$ dynes per sq.cm.; density of brass = 8·5 gm. per c.c.; density of air at S.T.P. = 1·30 gm. per litre. Standard atmospheric pressure = $1{\cdot}013 \times 10^6$ dynes per sq.cm.

Draw a diagram of the experimental arrangement and describe briefly how the experiment would be performed.

(London Univ. Inter. B.Sc.)

20. Define *stationary waves*, *node* and *antinode*.

Show with the aid of diagrams the variation with time of the particle velocity and the excess pressure at a point between a node and an antinode in an organ pipe sounding in its fundamental mode, and indicate the phase relation between them.

The air in a uniform tube of total length one metre is set in vibra-

tion by a tuning-fork held near the open end, while the length of the vibrating air column is slowly varied by a movable piston. Two adjacent resonant positions are observed for lengths of air column 34·9 cm. and 58·9 cm. respectively. Calculate the frequency of the fork, and find for what other lengths of column resonance may be observed. Assume the velocity of sound to be 34,000 cm. per sec.

(London Univ. Inter. B.Sc.)

21. Explain the terms *resonance* and *beats* as applied to sound, giving *one* illustrative example of each of these phenomena.

A vibrating tuning-fork, of frequency 330, is held over a long glass tube full of water which is allowed to run out slowly. Calculate the lengths of air column giving the first two positions of resonance if the velocity of sound is 33,000 cm./sec. and the diameter of the tube is 4·0 cm. (Cambridge H.S.C.)

22. Describe *two* experiments which show that sound is a wave motion.

What are standing (stationary) waves, and how do they differ from travelling waves?

A tuning-fork is held over the open end of a vertical tube fitted with a movable piston. The first and second resonances occur when the lengths of the tube are respectively 15·3 cm. and 48·3 cm. Explain these results, and determine the frequency of the fork. (Velocity of sound in air = 330 metres/sec.)

(Cambridge G.C.E. Advanced level.)

23. Describe the method and explain the theory of a practical determination of the frequency of a tuning-fork which does not assume the velocity of sound, or the frequency of another fork.

If the frequency of a tuning-fork is 550 cycles/sec., where will the first and second resonance positions be located in a resonance tube whose end correction is 3·5 cm.? Assume that the velocity of sound in air at 0° C. is 330 metres/sec. and that room temperature is 20° C.

(London Univ. G.C.E. Advanced level.)

24. Describe a method of determining the velocity of sound in coal-gas, assuming that the velocity in air at room temperature is known.

A Kundt's tube is fitted with a brass rod 5 ft. long, clamped at the centre. If the velocity of sound in brass is 11,500 ft./sec., find the frequency of the note produced when the rod is stroked longitudinally. Find also the distance between the heaps of powder in the tube, assuming that the velocity in the air contained in the tube is 1120 ft./sec. (Cambridge G.C.E. Schol. level.)

25. Describe the formation and properties of stationary wave motion.

Describe how a resonance tube and sources of sound of known frequencies may be used to determine the velocity of sound in air. Show how the velocity at 0° C. may be calculated from the value at room temperature. (Northern Univ. H.S.C.)

26. Explain the following terms: *amplitude, antinode, wave length* and *wave form* with reference to sound waves. Draw diagrams showing the positions of the nodes and antinodes in two organ pipes, one with one

end open and the other with both ends open, when they are sounding (i) their fundamentals, and (ii) their first overtones.

Describe *briefly* how you would show the presence of nodes in (*a*) a sounding organ pipe, (*b*) a vibrating plate and (*c*) a vibrating string.

(London Univ. G.C.E. Advanced level.)

27. The note which is emitted when the cork is rapidly withdrawn from an empty bottle is due to the air in the neck acting as a piston to the air in the bottle itself, which contracts and expands adiabatically as a whole.

Find the note at N.T.P. emitted from a Winchester quart bottle ($2\frac{1}{2}$ litres), the length and diameter of whose neck are each 2 cm. (Density of air at N.T.P. is 1·293 gm./litre : 1 atmosphere = 10^6 dynes/cm.^{2}; $\gamma = 1{\cdot}41$.) (Oxford Univ. Schol.)

28. What do you understand by *forced vibrations* and *resonance*?

The frequency (f) of vibration of the air in a Helmholtz resonator consisting of a vesse lof volume V having a cylindrical neck of length l and cross-sectional area A is given by

$$f = \frac{c}{2\pi}\sqrt{\frac{A}{lV}},$$

where c is the velocity of sound in the gas. Describe how you would verify experimentally this relation between resonating frequency and volume when A and l are constant.

What will be the frequency of the note emitted at room temperature (17° C.) when a cork is rapidly withdrawn from a wine bottle containing 2·5 litres of air? The length and diameter of the neck are 2·2 cm. and 2·0 cm. respectively.

(London Univ. G.C.E. Advanced level.)

29. Discuss the manner of vibration of a simple tuning-fork and the nature of its sound field. Describe any desirable modifications for its use as a frequency standard.

Assuming that the frequency of a tuning-fork depends on (*a*) its linear dimensions l, (*b*) the density ϱ and (*c*) the Young's modulus E, of the material of the fork, derive by the method of dimensions, the relationship between the frequency and the variables, l, ϱ and E.

If both E and the linear expansion of steel vary linearly with temperature, deduce the percentage change of frequency of a steel fork per deg. C. rise in temperature.

(Temperature coefficient of linear expansion of steel = 12×10^{-6} per deg. C. Temperature coefficient of Young's modulus of steel = -24×10^{-5} per deg. C.)

(London Univ. G.C.E. Schol. level.)

CHAPTER 12

APPLIED ACOUSTICS

12.1 Character of Musical Sounds

Musicians are commonly heard to talk of the *quality* or *timbre* of a note, meaning thereby the characteristics which enable two notes from different sources to be distinguished, even though they are of the same pitch; for example, a note of pitch middle C sung by a female voice can readily be told apart from the same note played on a piano because the two notes have different *qualities*.

Each particular source produces a note of characteristic quality for two main reasons; firstly, it was seen in the previous chapter that any vibratory system—a stretched string, for example—can oscillate in a fundamental mode and in a range of harmonic modes. It is quite possible for the string to vibrate in several of these modes at the same time, thus the sound to which it gives rise is not a pure note of one frequency but a complex tone made up of a fundamental and a few selected harmonics.

The particular range of harmonics depends not only on the vibratory system (it has been seen that some systems produce a complete range of harmonics while others can produce only the odd harmonics), but also on the way in which it is excited. Consider, for example, a stretched string struck by a padded hammer as in the action of a piano. This will excite all harmonic modes of which the string is capable (a complete range, see page 329) *except* those modes which have a displacement node at the point of impact, for in each of these modes the string should be at rest at this point.

If the string is struck at the mid-point, then the 2nd, 4th, 6th, etc., harmonics will not be sounded, but the fundamental and all odd harmonics will respond; if the point of impact is moved to a position one-third of the way along the string, the 3rd, 6th, 9th, etc., harmonics will not be sounded, but all the others will be present.

In cases such as these, the complex note is said to consist of a fundamental and a series of *overtones*, the second harmonic being called the first overtone and so on.

The ear is very good at recognising the fundamental and range of overtones to be expected from any given source, and uses this as one piece of information in identifying the source.

The actual range of frequencies heard by the ear is complicated by an effect described as 'combination-tones'. If two frequencies are heard at the same time, due to certain conditions in the auditory system, the

brain is aware of the two frequencies and frequencies equal to the sum and difference of the two original frequencies; thus if a fundamental f cps and its third harmonic $3f$ cps are sounded together, the difference frequency $2f$ cps and the sum frequency $4f$ cps are also heard.

This fact, together with the ability of the ear to recognise a harmonic series, can be used to deceive the ear; an experiment to illustrate this may be performed as follows. Two sources of 400 cps and 600 cps are sounded together and then an observer is asked to adjust another source to be in tune with the sound he hears; most people confidently tune the source to 200 cps. It seems as though the mental reasoning (carried out instinctively) is as follows: 'The fundamental cannot be 400 cps, for 600 cps is not a harmonic of this frequency, but the two notes, together with the difference and sum frequencies of 200 cps and 1000 cps do form a harmonic range based on 200 cps, therefore the fundamental note must be of frequency 200 cps. If a further note is added at 500 cps, the subject then 'hears' a note of 100 cps—the student can probably see for himself the reason for this.

This device is sometimes used by the builders of organs—the lowest pedal note needed on an organ has a frequency of about 16 cps, which could be produced by a pipe about 32 ft long; if space does not permit this, the organ builder provides one pipe 16 ft long sounding at 32 cps and another 10 ft long sounding at 48 cps. When these are sounded together, the hearer interprets the resultant sound as a fundamental of 16 cps with overtones at 32, 48 and 80 cps.

The second factor which determines the character of the note produced by a source of sound is the manner in which the source is excited into vibration. Consider the stretched string; this may either be bowed as in the violin, plucked as in the harp, or struck with a padded hammer as in the piano; the waveform of the sound produced in each of these cases show that at the start there is a region where the waveform is confused and not periodic. Such a waveform is called a *transient*, and it can be shown by methods beyond the standard of this book that a transient is equivalent to a wide band of frequencies, not a harmonically related series. Thus a sound starts as such a band of frequencies, most of which quickly die away leaving the fundamental of the source and its overtones. The growth and die-away of the transient is often described as the *attack* of the note and this also lends character to a sound; in fact it is probably the major factor by which a source of sound is recognised.

12.2 Musical Instruments

It would be out of place in a book of this standard to discuss the instruments of the orchestra in great detail, but a few selected ones will be described in so far as they offer illustrations of the work in the preceding chapters.

Broadly speaking, instruments can be divided into three classes: stringed instruments, in which the sound is generated by a vibrating string; wind instruments, which depend on a vibrating column of air; and, lastly, percussion instruments, in which diaphragms, bars or tubes are struck with hammers.

(a) Stringed Instruments

Within this family of instruments there is a further subdivision, for the string may be either bowed, plucked, or struck with a hammer; all stringed instruments show one common feature, the vibrating string is so small that it cannot cause much disturbance in the surrounding air, hence all stringed instruments are provided with sounding boards. This string is mounted so that its vibration is handed on to the sounding board and this in turn causes a large amount of air to vibrate.

(i) Bowed Strings

The violin, viola, 'cello and double-bass all come within this class; apart from size, they are similar in construction, therefore only the violin will be described in detail.

The strings are stretched over a bridge as shown in Fig. 12.1 The bow, made of horsehair and well impregnated with resin, is drawn across the strings, and sets them into transverse vibration. The action

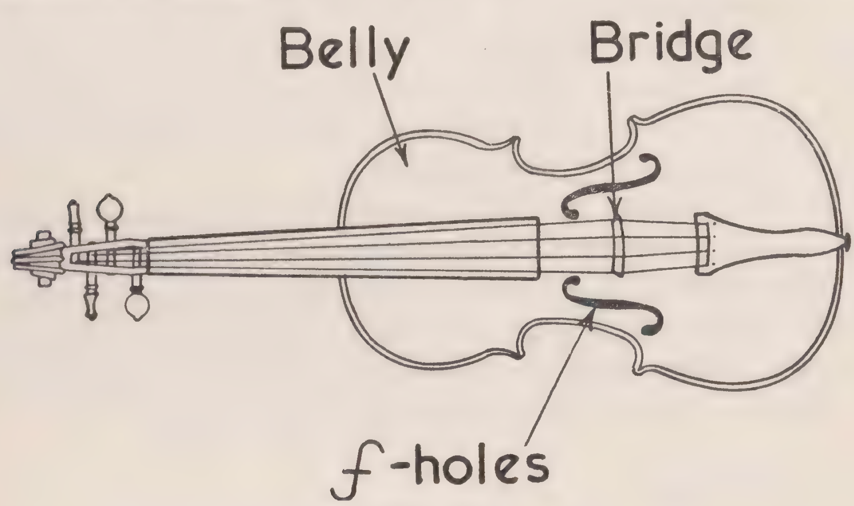


Fig. 12.1

of the bow has been explained in many ways; a plausible account is as follows. Friction between bow and string causes the string to be drawn aside by the bow, the restoring force on the string gradually increases until it overcomes limiting friction, whereupon the string flies back. The natural vibration of the string carries it on past the rest position to an extreme displacement in the opposite direction against the motion of the bow (which is now only able to apply the much smaller forces of sliding friction). At this point the string again starts to move in the same direction as the bow and comes to rest relative to it, static

friction now applies and the string is once more drawn aside, only to repeat the process.

The vibrations of the strings are passed by the bridge to the belly of the violin and thence by a sound-post to the back. These two surfaces act as the sound-board, also the air enclosed within the belly acts as a Helmholtz resonator, broadly tuned by means of the *f*-holes.

The belly has certain flat-plate modes of vibration of its own; modern work has shown that one factor contributing to the magnificent tone of violins made by such people as Stradivarius is the thinness and flexibility of the belly. This allows it to vibrate freely in several modes. Normally, however, a bass-bar is glued beneath the belly, giving it stiffness in one direction and ensuring that it has a mode of vibration comparable with that of the lowest frequencies sounded by the strings.

The strings are made of gut or wire; they are all of the same length and they are all adjusted to roughly the same tension so that they may present the same 'touch' to the violinist. Slight variation in tension is used to give a fine control over the tuning. The rough tuning of each of the four strings is achieved by making them of different mass per unit length. Thus the string of highest pitch is of very fine wire or gut, the intermediate strings of thicker gut, while the bass string is bound with very fine copper wire to increase its mass.

To play the violin, the string is pressed against the finger-board by a finger of the left hand, this is described as 'stopping the string'. It effectively shortens the vibrating length and so increases the natural frequency of the string. The violinist is at liberty to stop the string where he pleases and so may make any fine adjustments of pitch that he feels are necessary (see Section 12.3).

Normally the strings of a violin vibrate in their fundamental mode, but if the violinist touches a string lightly with a finger whilst bowing, then the point touched becomes a node and the string sounds in a harmonic mode. Thus touching the string at the mid-point would make it vibrate in two half-wavelength loops and it would sound in the second harmonic mode, or one octave above the open string.

(ii) Plucked Strings

In this category come the harp, the harpsichord, guitar, banjo and many others.

Plucking the strings produces a particular quality of sound due to the initial transient. Also the note, once struck, cannot be maintained but dies away; this is countered in the mandolin and similar instruments by repetitive plucking.

(iii) Struck Strings

The piano provides an example of this class of instrument.

One string is provided for each note, although better instruments

provide two or three strings for each note at the treble end of the range; the strings are of steel wire (bound with copper wire for the lowest notes), and are stretched on a steel frame which is rigidly bolted to a wooden sounding-board. The strings are all stretched to about the same tension, the pitch being determined by length; the bass strings are several feet long and the treble ones only a few inches, but the final tuning is done by slight variation in the tension.

Depressing the key operates a complicated chain of levers, these lift a small felt pad off the string an instant before it is struck by a padded hammer; after striking the string once, the hammer is held clear of the vibrating string. When the key is released, the felt pad, called a damper, is pressed against the string once more and quickly brings it to rest. If the sustaining (or 'loud') pedal is pressed, the dampers are held away from the strings, and thus the sound can be maintained after the key is released.

The hammer strikes the string at about an eighth of its length from one end, the position having been selected by long experience as that which energises the most acceptable range of overtones. Other strings in the piano vibrate in sympathy with the ones which are sounding; normally this reinforces the harmonic frequencies, for the unused strings are prevented from sounding in their fundamental modes by the dampers pressed against them, thus a note with a prominent range of overtones is produced. This is one of the effects contributing to the richness of tone in the piano.

It should be noted that, unlike the violin, the tuning of the piano is fixed, i.e. once the piano is tuned, the pianist cannot alter the pitch of the individual notes.

(b) Wind Instruments

Once again several broad groups of instruments can be recognised; firstly, those in which an air column is caused to resonate by a jet of air blowing over an aperture or past an edge, sometimes called flue pipes; and, secondly, those in which the energising element is a vibrating reed. Before discussing the instruments it will be beneficial to examine the action of a jet of air or a reed as a source of sound.

(i) Jet Tones and Edge Tones

It was mentioned in Chapter 9 that if a fluid flows past an obstacle, the motion becomes turbulent if the fluid velocity exceeds a certain critical value. The turbulence involves a series of vortices which are formed alternately on opposite sides of the obstacle and then swept along by the stream, Fig. 12.2 (*a*). These vortices account for the fluttering of flags (Fig. 12.2 (*b*)) and the 'singing' of telephone wires in a wind.

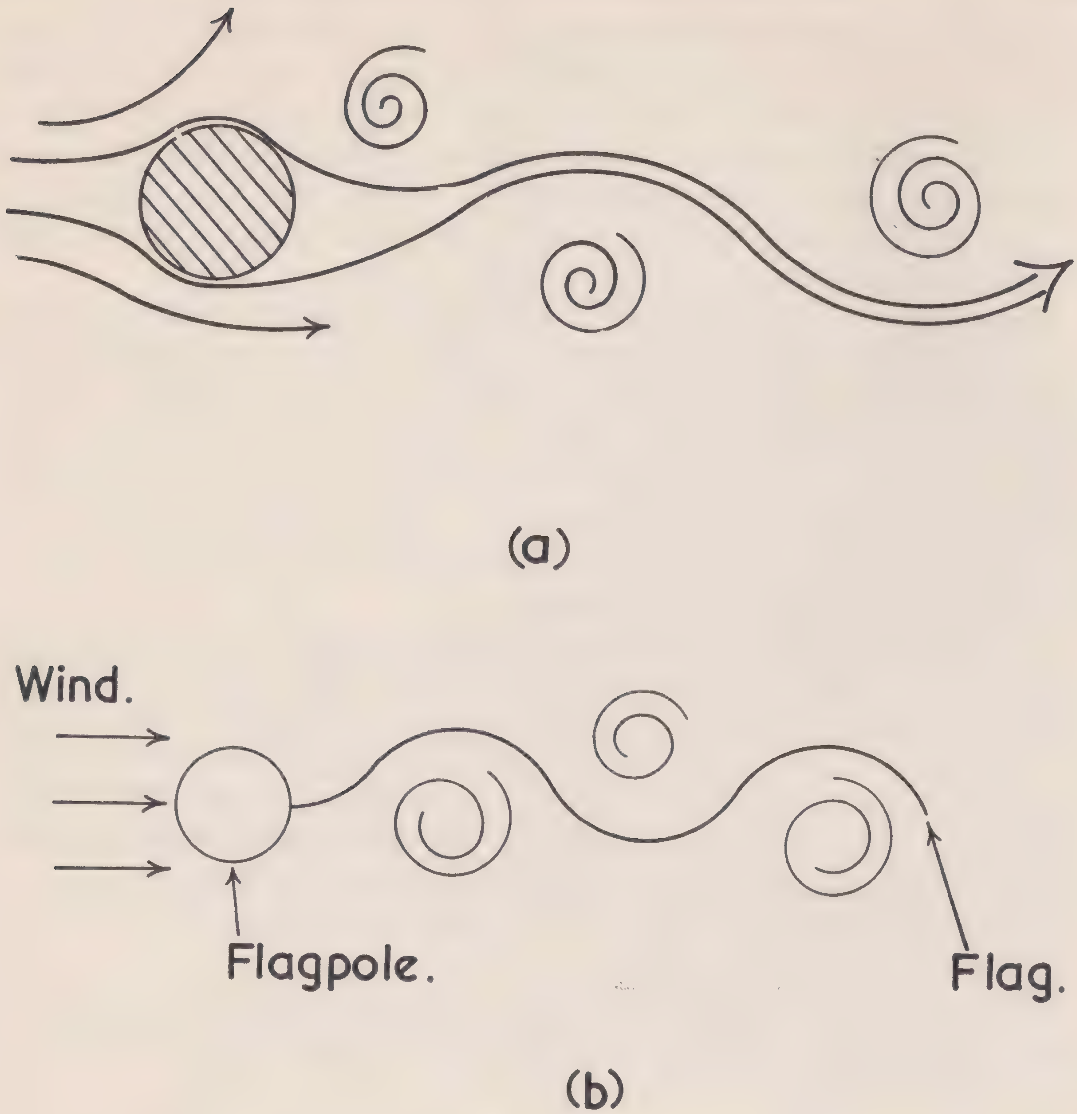


Fig. 12.2

If a jet of air issues from an orifice, vortices are formed in a similar manner and give rise to feeble sound waves in the surrounding air. Purse the lips and blow—a high-pitched noise will be heard caused by the vortices, now use the tongue to adjust the volume of the mouth so as to form a resonant cavity (Helmholtz resonator) and the jet will produce a musical note at the natural frequency of the cavity—more familiarly, this process is called whistling.

The frequency of the *jet tone* depends on the velocity of the air stream amongst other factors, consequently blowing harder will give a higher note; but in any case, the frequency is not very well defined unless the jet can be used to energise some very powerful resonator.

The 'sensitive flame' is another example of this motion; if gas issues from a jet at a low velocity, vortices are not formed and the gas burns with a long steady flame, but if the pressure is increased, the flame becomes irregular and noisy. Such a flame can be used to detect sound waves in air by adjusting the gas pressure so that the flame is on the

point of becoming unstable in still air; a strong sound wave is enough to upset this condition and the flame changes from its steady mode to the irregular condition; this could be used to detect the nodes and anti-nodes of a standing waveform.

The stability of the frequency of a jet-tone is improved if the jet is directed against a knife-edge. This can be demonstrated by blowing as above when an indiscriminate noise is heard, but place the edge of a piece of paper in the air stream and the noise turns to a high-pitched note. The frequency of the note increases as the edge moves nearer the jet—this can again be demonstrated with the lips and a piece of paper.

A note produced in this fashion is called an *edge tone.*

(ii) Flue Pipes

Many of the pipes used in an organ are flue pipes, and in addition all the members of the flute family (flute, piccolo, fife) belong to this class, as does the recorder and the 'tin whistle'.

The construction of the organ pipe is shown in Fig. 12.3 (*a*) and the recorder in Fig. 12.3 (*b*) ; it will be seen that they are substantially the same. In each a narrow jet of air is directed against a sharp edge and so produces edge tones. The flute is a cylindrical tube closed at one end and has a hole bored in the side near this end; the instrument is held crosswise and a narrow jet of air directed from the lips of the performer to strike the far edge of the hole, once again giving rise to edge tones (see Fig. 12.3 (*c*)).

It should be noted that in the case of the organ pipe, and similarly in the other instruments, the lower lip and languid are so shaped that the majority of the air stream comes out of the mouth of the pipe, very little passes into the pipe itself. The production of vortices at the upper lip however causes the air to vibrate in and out of the mouth of the pipe. This excites the air in the pipe into resonant vibration if the edge tone and the air column in the pipe have approximately the same natural frequency.

The bottom end of the pipe is effectively open at the mouth, consequently the pipe resonates as though open at both ends, i.e. the displacement standing wave is as shown in Fig. 12.4 (*a*). Thus the pipe produces a note of wavelength λ in air where λ is given by

$$\lambda/2 = l \qquad (1)$$

which leads to

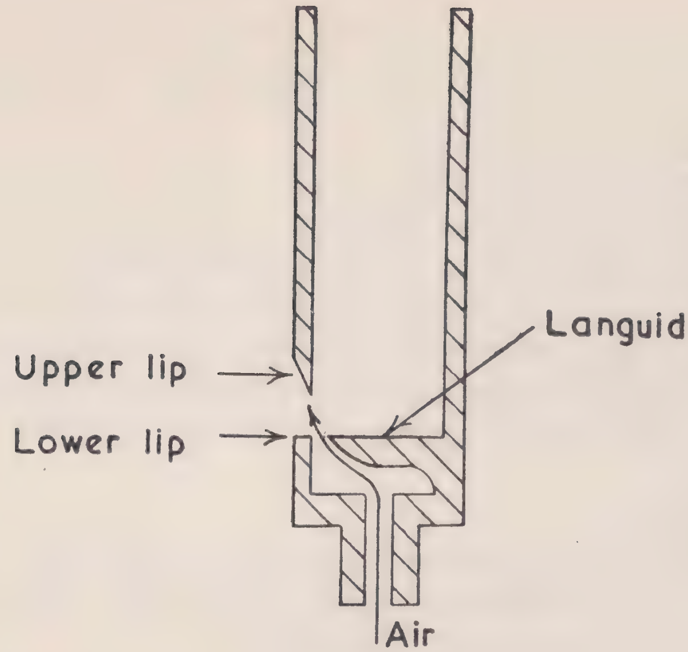
$$f = v/2l \qquad (2)$$

(see Section 11.10 (f)).

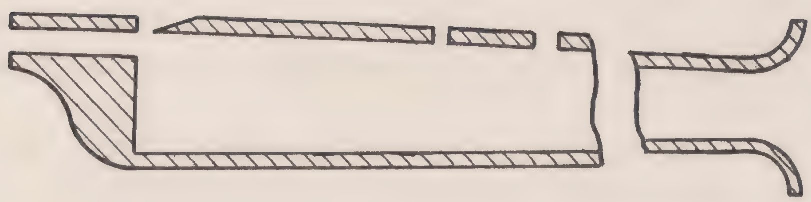
If the air in the tube changes temperature, the velocity of sound changes according to the law

$$v_t = 332\ (1 + t/546)\ \text{m.sec}^{-1} \qquad (3)$$

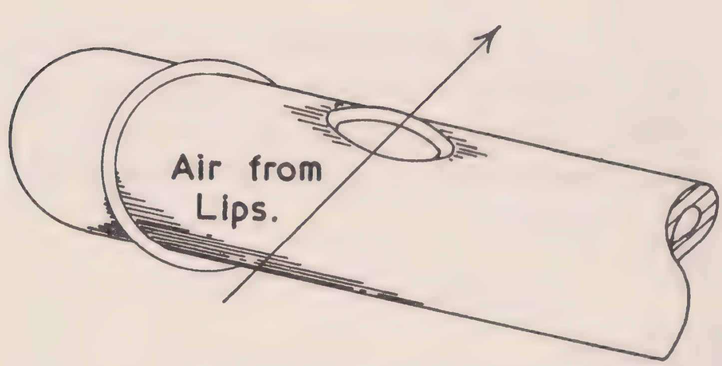
where t is the centigrade temperature of the air.



(a) Organ Pipe.



(b) Recorder.



(c) Mouthpiece of flute.

Fig. 12.3

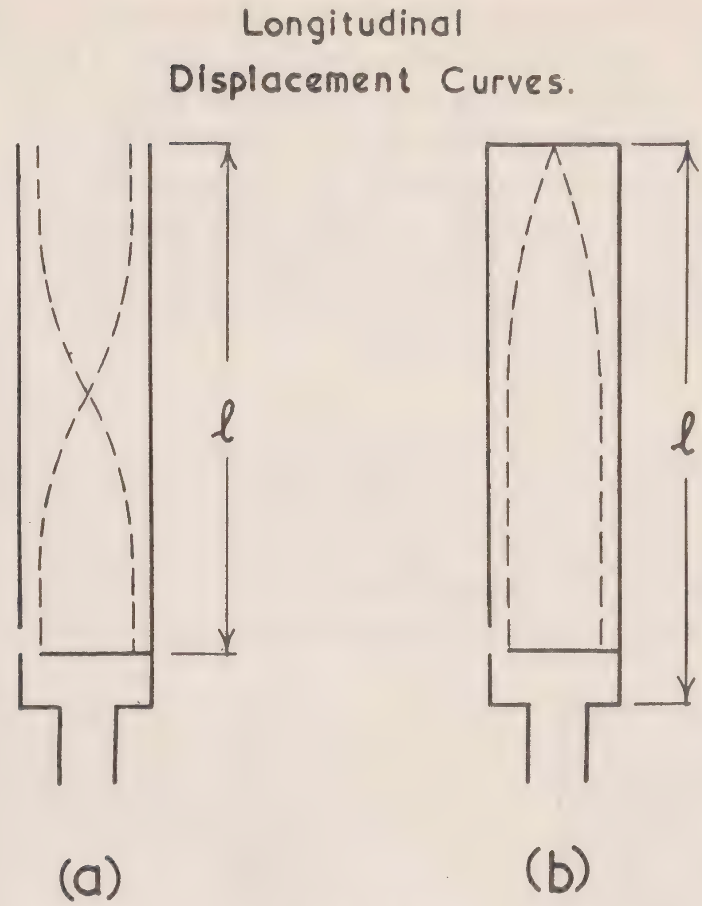


Fig. 12.4

Thus the frequency may be written as

$$f_t = (166/l)\ (1 + t/546) \text{ cps} \quad . \quad . \quad . \quad (4)$$

(l measured in metres), from which it is seen that the frequency rises with temperature.

This is a disadvantage of all pipe instruments, and especially so for those blown by mouth, when the hot moist breath of the player causes the frequency of the instrument to rise appreciably during the first few minutes of use.

It was seen in the previous chapter that a pipe open at both ends could resonate to a full harmonic range; this is true of the open organ pipe, which produces a full rich tone, characteristic of a 'church organ'.

The organ pipe may also be stopped at the upper end; in this case the displacement standing wave is as shown in Fig. 12.4 (b) and we have:

$$\lambda/4 = l$$

leading to

$$f = v/4l \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This is half the frequency produced by the same pipe when open, thus stopping a pipe lowers its pitch by an octave.

In this case the pipe resonates to the odd harmonics only (see Section 11.10 (b), page 338) and the tone loses some of its body.

The flute and the recorder have a series of holes bored in the pipe; the holes are normally covered by the fingers or by pads operated by keys. If the lowest one is opened, the displacement antinode moves from the end of the pipe to the hole, thus shortening the effective length of the pipe and so raising the pitch of its resonant frequency. A series of holes bored along the length of the pipe enables the instrument to play the notes of the scale.

(iii) Reeds

Basically, a reed consists of a strip of flexible material mounted over an aperture in a flat plate as in Fig. 12.5 (*a*). The reed normally lies flat as in Fig. 12.5 (*b*), but if a stream of air is directed against it, the pressure forces the reed through the window as in Fig. 12.5 (c). The air can now flow freely through the gap, therefore the pressure on the reed

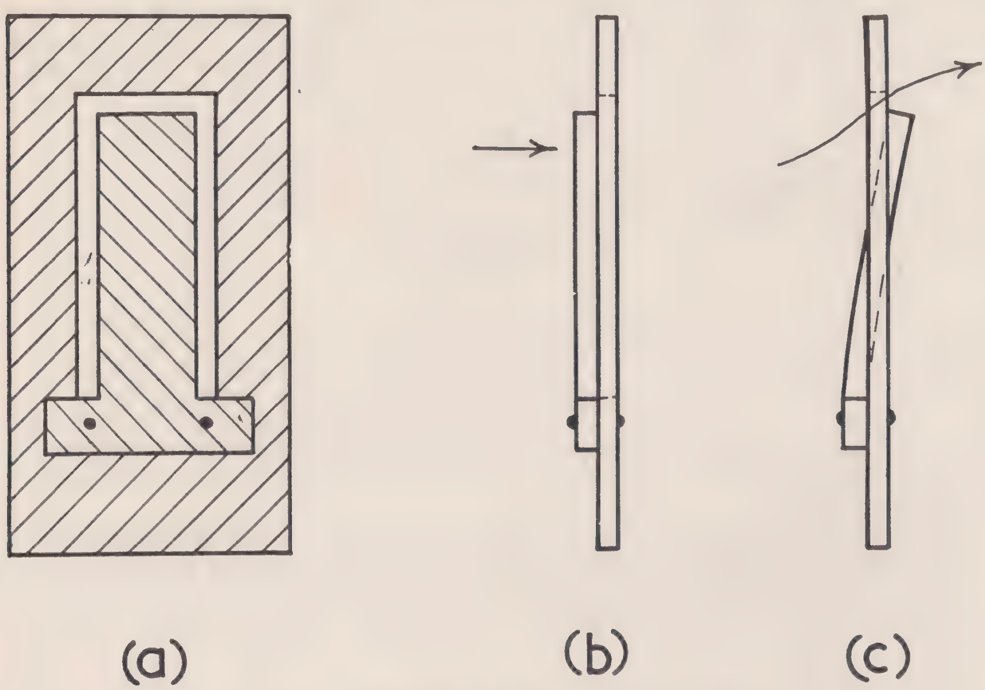


Fig. 12.5

decreases and it springs back to its original position, whereupon it performs the motion again at its own natural frequency. The air passing through the reed is therefore chopped into pressure pulses at the frequency of vibration of the reed.

The reed described above is called a *free reed*, since it performs an uninterrupted motion. In another variety, the reed is slightly larger than the window and when at rest curves away from it. The action of the air causes the reed to press down on to the window and close it for the passage of air; this is called a *beating reed*.

Again, some instruments use double reeds; these consist of two flat strips of cane fixed to the end of a tube, these act as two beating reeds.

Finally some instruments require the player to press his lips together and use them as double reeds.

(iv) Reed Instruments

The reed emits quite a loud sound by itself, due to the stream of pressure pulses which it generates. Hence some instruments use only a vibrating reed as the source—the mouth-organ, concertina and harmonium are examples, all of which use free reeds. Other instruments use the reed to energise a resonant air column. In this class we have the oboe and bassoon, using double cane reeds; the clarinet and saxophone which use single beating reeds; and the organ which contains pipes energised by either free or beating reeds.

The note emitted by a reed contains a very complex range of overtones and the sound is rather coarse, the free reed being worse in this respect than the beating reed. The use of a resonant pipe modifies the tone somewhat, nevertheless quite a wide range of harmonics still persists in most reed instruments .

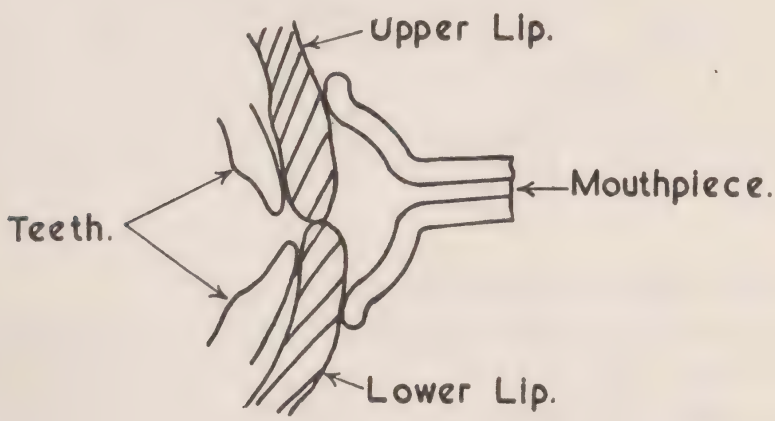


Fig. 12.6

Finally, in the family of musical instruments known as 'the brass' (i.e. trumpet, trombone, tuba, horn, cornet, euphonium, etc.), the lips of the player are used as a double reed as illustrated in Fig. 12.6. The resonating air column is contained within a narrow bore metal tube, suitably coiled so as to be manageable by the player, and flared out to a bell at the far end. Such a resonant column is capable of vibrating in a full harmonic range of modes, but most brass instruments cannot be excited in the fundamental mode. The lowest note is that due to the second harmonic, thus if the instrument is 'overblown' (that is, made to sound in higher harmonic modes), it produces a series of notes whose frequencies are in the ratio of 2 : 3 : 4 : 5 : 6. In the tonic sol-fa system of musical notation, these notes would be called doh, soh, doh',

me′, soh′. They are the notes which can be played by a bugle or coaching-horn.

Orchestral instruments have the same range of open notes, but are also provided with valves (two, three or four in number according to the instrument). When a valve is pressed, a small length of tube is added to the resonant column and so the pitch of the note is flattened; by this means the gaps between the open notes can be bridged and a full scale played.

An exception to the above is met in the trombone; in this instrument one convolution of the tubing is formed by a U-shaped piece of tubing, the arms of which slide within two others; by moving this piece (called the slide) in or out, the total length of tubing can be altered and so the pitch of the note adjusted.

(c) Percussion Instruments

This section includes all the instruments which are sounded by being struck or beaten; the range includes bells, drums, xylophones, tambourines, castanets, etc. Some of these instruments (such as the castanets) are not tuned, but give merely a sharp percussive noise and are used to point the rhythm of a piece of music. Other percussion instruments are tuned and can be used for melodic purposes, the xylophone provides an example of this; it consists of a series of short wooden bars supported near the ends and struck at the mid-point; the length of the bar is the chief factor in determining the pitch of the note emitted, but a metal resonator tube is fitted underneath each bar to emphasise the fundamental.

12.3 Musical Intervals and Scales

When two notes are sounded together, it is found that there are some pairs which blend very well and produce a pleasing sound, while other pairs give a most discordant and irritating noise: different nations appear to have various tastes in this respect and the discussion given here is based on the music of the European nations. Many of our musical instruments, such as the piano, have notes of fixed tuning. The notes used are those which have been found by experience to produce the maximum number of concordant pairs. Such a collection of notes is said to constitute a *musical scale.*

The simplest scale in our music is called the *scale of C major*, and it consists of eight notes whose pitch would be described by a musician as:

C D E F G A B C′

The upper note C′ of this series is the octave of the lower one.

The octave interval is found by the majority of people to give the most concordant pair of notes; these notes have frequencies in the ratio 2 : 1 and it seems that simple arithmetic ratios (such as 2 : 1,

3 : 2, etc.), between the frequencies lead to pleasing sensations, while 'difficult' ratios, i.e. 119 : 241 give dissonance. With this in mind, the frequencies of the middle notes of the scale are fixed in simple ratios to each other, as shown below:

C		D		E		F		G		A		B		C′
	8 : 9		9 : 10		15 : 16		8 : 9		9 : 10		8 : 9		15 : 16	

If we choose a frequency for the lower C, those of all the other notes of the scale can be worked out; 240 cps is a convenient number, for then fractions are avoided and the scale becomes:

Pitch . .	C	D	E	F	G	A	B	C′
Frequency .	240	270	300	320	360	400	450	480 cps

It will be seen that a number of concordant intervals can be chosen from this scale, for example G to C, B to E, or C′ to F are all of ratio 3: 2 while F to C, G to D, A to E, or C′ to G of ratio 4 : 3 and so on.

Three intervals appear in the scale above, i.e. 8 : 9, 9 : 10 and 15 : 16, these are called a *major tone, minor tone* and *semi-tone* respectively.

If we construct a similar scale, but starting at G (this is the scale of G major) the frequencies will be as given below (notice that if a scale is continued up into the second octave, the frequency of any note is double that of the corresponding note in the lower octave):

C Major .	C	D	E	F	G	A	B	C′	D′	E′	F′	G′
	240	270	300	320	360	400	450	480	540	600	640	720
G Major .					360	405	450	480	540	600	675	720
						8:9	9:10	15:16	8:9	9:10	8:9	15:16

It will be seen that the frequencies required for this scale are the same as those needed for C major with two exceptions, the 'near-miss' which occurs at A (405 for 400) and the 'bad-miss' of 675 which comes somewhere between F′ and G′. To accommodate this latter case we add an extra note called 'F sharp' and written as F♯. Thus, apart from the difficulty of the frequency of the note A, the scale of G Major is:

G A B C D E F♯ G′

It will be seen that the interval from F♯ to G′ is a semitone (ratio 15 : 16) but that the interval from F to F♯ is 128 : 135, which is a little less than a semitone and is called a *chromatic semitone*. If we continue to form further scales, starting on each of the different notes, we find that we have to add further semitones at C♯, D♯, G♯ and A♯. Since some of these intervals split minor tones, instead of the major tone (F to G) discussed above, then there will be two sizes of chromatic semitones.

We now have the octave divided into twelve semitones:

C C♯ D D♯ E F F♯ G G♯ A A♯ B C′

but the semitones are of three different sizes.

All of this, however, does not resolve the difficulty of the frequency of the note A (i.e. 400 in the scale of C major but 405 in G major). In addition we find that as further scales are formed, all the notes (including the 'sharps') have to be altered slightly in frequency for one scale or another. The solution adopted on modern keyed instruments is to tune each note to a compromise frequency, roughly the average of the frequencies needed to make it fit into all scales. The octave is then divided into twelve *equal* semi-tones of ratio $1 : \sqrt[12]{2}$, and the notes are said to form a scale of *Equal Temperament.*

The result of this is that no scale corresponds exactly to the mathematically accurate scale (or *just intonation*). For example the scale of C major becomes:

	Pitch . . .	C	D	E	F	G	A	B	C
Frequency	Just Intonation .	240	270	300	320	360	400	450	480
	Equal Temperament	240	269·4	302·4	320·4	359·6	403·6	453·0	480

Moreover, the mistuning of each scale is different; the trained ear can recognise the small differences between scales and can tell if a piece of music written and well known in one scale is being played in another. A scale adjusted in this fashion obviously only applies to keyed instruments where the tuning of the notes is fixed—a violinist makes the notes by stopping the strings with his fingers at any point he chooses and thus can play every scale in just intonation.

12.4 Transcription of Sound

One of the most powerful factors in the modern trend of commercial and domestic life is the growing ease and speed of communication with other parts of the world. This has been brought about by the advent of such instruments as the microphone, whereby a sound wave is converted into an electrical waveform which can be amplified and transmitted to a distant point by telephone line or radio. At the receiving end the signal can be reproduced as a sound wave by either a loudspeaker or headphones.

Alternatively the electrical waveform can be used to make a 'record' which may be played back when required.

The student should have an idea of the acoustic principles involved in the design of instruments used for this purpose; but many of them rely on electrical effects, a description of which will be found in books on electricity or radio.

(a) The Microphone

Of the many sorts of microphone in common use, three may be singled out for description; *moving-coil* and *crystal* microphones are used mainly for broadcast and recording purposes, while *carbon granule* microphones are used by the G.P.O. in telephone handsets. The overall effect of each of these microphones is the same, to convert sound

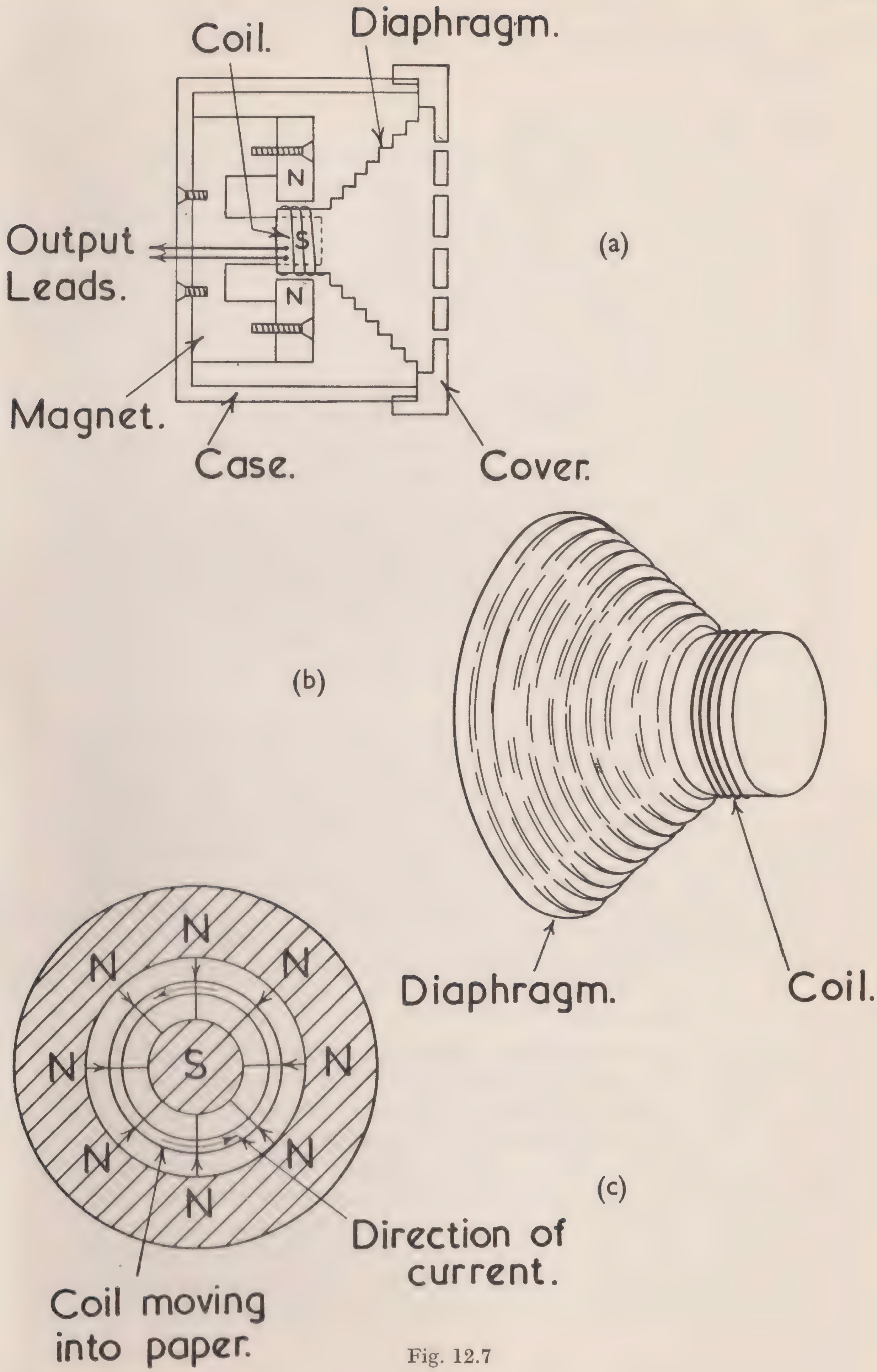

Coil.
Diaphragm.
N
S
N
Output Leads.
Magnet.
Case.
Cover.
(a)
(b)
Diaphragm.
Coil.
N
N
N
N
S
N
N
N
N
(c)
Direction of current.
Coil moving into paper.

Fig. 12.7

into electrical waveforms, but the mode of operation of each is rather different.

(i) Moving-coil Microphone

The construction of the microphone is shown in Fig. 12.7 (*a*).

A small, circular diaphragm, Fig. 12.7 (*b*), shaped like a shallow cone, is clamped to a case round its outer rim by a perforated cover. The diaphragm is made either of stiff card or fibre, or very thin metal foil and has circular corrugations pressed into it. This makes it very springy, and although it is clamped around the edge, a very slight pressure at the centre can move it a small distance in or out. Sound waves entering through the perforations of the cover provide this pressure, and if the diaphragm is very light and flexible, its motion is a more or less faithful reproduction of the sound waveform.

The rear side of the diaphragm carries a light cylindrical former of insulating material, on which is wound a coil having many turns of very fine wire. This former and coil fit into the gap of an annular magnet, Fig. 12.7 (*c*), the gap is wide enough to allow the coil to move freely. When a sound wave impinges on the diaphragm, the coil vibrates to and fro in a very strong magnetic field, and this induces a voltage in the coil. The magnitude of the voltage is proportional to the speed of the coil and the polarity depends on the direction in which the coil moves, so that a vibratory motion of the coil will produce an output voltage of oscillatory waveform.

The diaphragm, unfortunately, has a natural resonant frequency of its own, and thus tends to give a much greater output at this frequency than at others. Also the air cavities on either side of the diaphragm may have resonances. Further, due to various electrical effects, the output tends to fall at the higher frequencies. But the skilful designer can offset these two effects; for if the resonant frequency of the diaphragm is made very high, the enhanced response due to resonance can be made to compensate the high frequency fall in response, and the moving-coil microphone is capable of very faithful reproduction.

(ii) Crystal Microphone

The action of this microphone depends on a property possessed by some crystals and called the *Piezo-electric Effect*; crystals showing this effect produce a potential difference between opposite faces when subjected to mechanical deformation such as squeezing, bending or twisting. The magnitude of the voltage produced is proportional to the extent of the deformation and the polarity reverses if the deformation is changed in direction. Thus if a crystal produces a positive voltage when twisted one way, twisting it the other way will give a negative voltage. Many crystals show this effect, but those in most common use are Rochelle salt for low frequencies (i.e. in the audible range) and

quartz for high frequencies from just above the audible range up to many megacycles per second.

The crystal microphone uses a crystal of this sort. The diaphragm is connected to one end of a thin crystal, the other end of which is rigidly fixed to the microphone housing (Fig. 12.8). Vibration of the diaphragm bends the crystal to and fro and so generates an alternating voltage which is picked up by metal-foil electrodes stuck to the crystal as shown in the diagram.

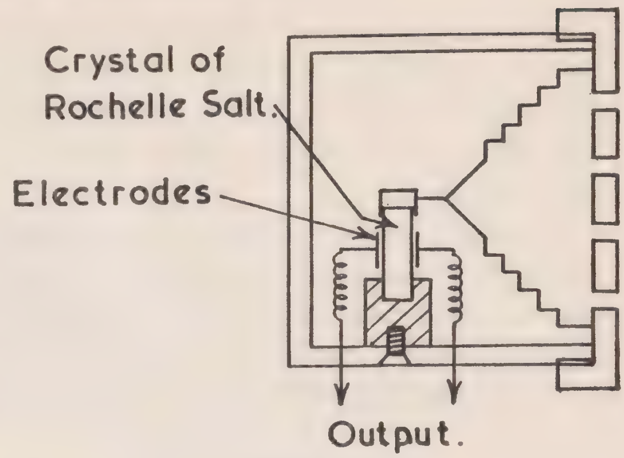


Fig. 12.8

The crystal microphone suffers from mechanical resonances of the diaphragm and, to a smaller extent, of the crystal; in addition, the output falls off very rapidly at higher frequencies. A further adverse factor is that the electrical waveform generated by a crystal is not a faithful copy of the waveform of the mechanical deformation applied to the crystal. Against this must be set the fact that a crystal microphone gives a very large output and needs less subsequent amplification than other types.

(iii) The Carbon-granule Microphone

Carbon in the solid form is a fairly good conductor of electricity. If, however, the carbon is ground down to a very fine powder and placed in an insulating tube, its ability to conduct electricity along the tube is diminished. This is probably because the granules have only

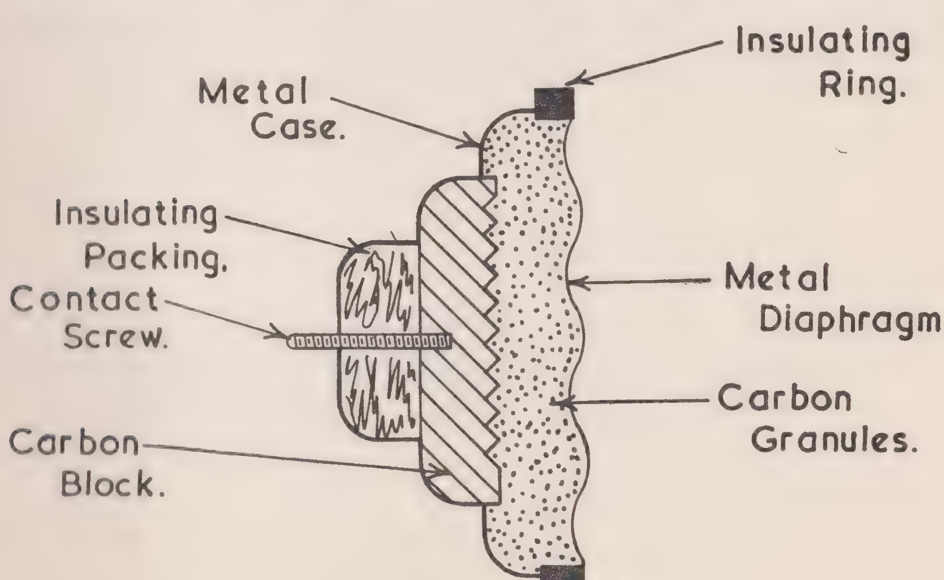


Fig. 12.9

small areas of contact with each other and the path for the flow of current is badly restricted. If the granules are compressed, then the contacts between individual grains will be increased and the current flow will also be improved; this is the basis of the carbon-granule microphone.

The microphone capsule is shown in Fig. 12.9, from which its construction is evident; electrical connections are made to the contact screw on the case and to the metallic diaphragm. Sound waves make the diaphragm move to and fro, which alternately increases and decreases the pressure on the carbon granules.

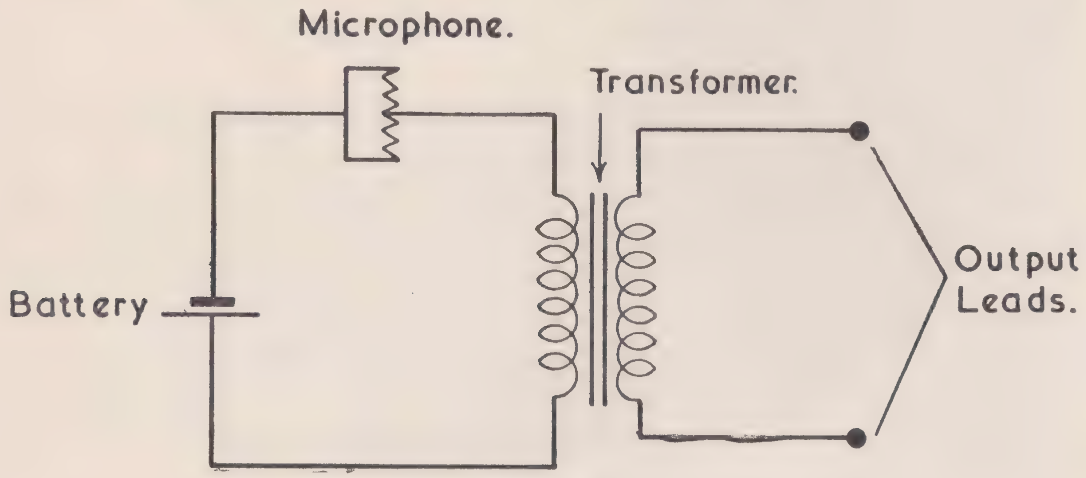


Fig. 12.10

The microphone is connected in circuit as shown in Fig. 12.10. A battery passes a small continuous current through the microphone and the primary winding P of a transformer. If the microphone is screened from sound, this current will be of constant magnitude (shown in the region OA of the graph in Fig. 12.11). Nothing will appear in the secondary winding of the transformer, for it produces an output only when the current through the primary changes.

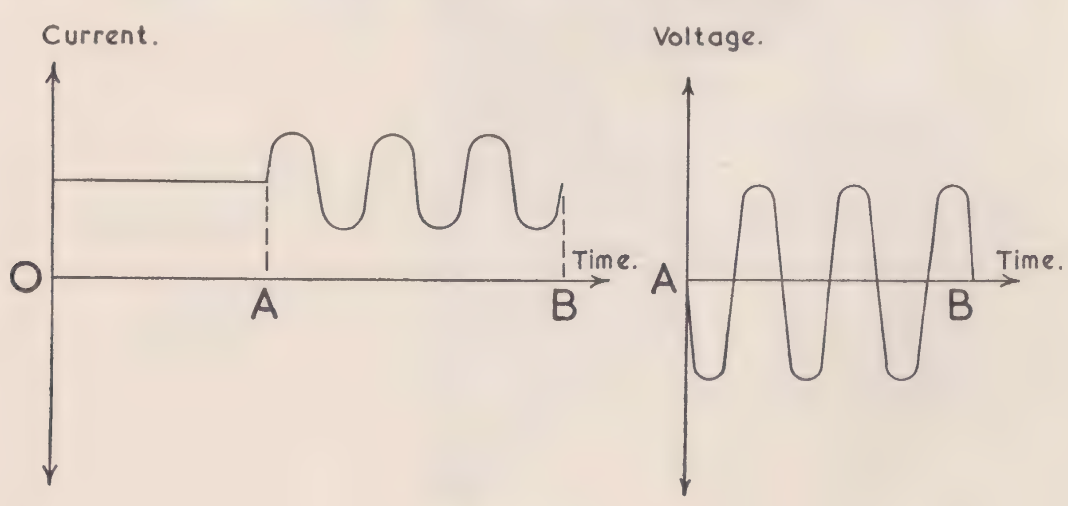


Fig. 12.11

If the microphone is allowed to receive a sound wave, then the motion of the diaphragm will vary the pressure on the granules and so the current through the microphone will fluctuate; this is shown in the region *AB* of the graph. The changing current produces an alternating voltage across the secondary windings of the transformer.

(iv) Comparison of Microphones

The main features one looks for in assessing the performance of a microphone are:

1. A large electrical output.
2. Faithful reproduction up to high frequencies.
3. Absence of background noise.
4. Convenience in use.

The moving-coil microphone does not give a very large output, but modern designs of the instrument give a very faithful reproduction of the sound wave up to the highest audio-frequencies. The crystal microphone gives a large output but unfortunately generates some spurious harmonics, also its output falls at higher frequencies.

The carbon microphone gives a large output with good frequency response, but generates a background hiss (like the noise of frying sausages), said to be due to small arcs occurring between the granules; it also needs a battery and this is rather inconvenient. The current to energise the microphone in a telephone is usually provided over the telephone wires from a central battery at the telephone exchange.

(b) The Loudspeaker

In diagrammatic form, the loudspeaker is similar to the moving-coil microphone. It is used in the opposite sense, i.e. an alternating current is fed into the coil (called the *speech coil* in this case). The current interacts with the field of the magnet, producing a vibratory motion of the coil; this in turn moves the diaphragm back and forth and generates sound waves in the air.

In practice the loudspeaker is much larger than the moving-coil microphone. In the latter case the diaphragm must be very small and light so that the feeble pressure in a sound wave may produce a reasonable motion; but a large current in the speech coil of a loudspeaker can be used to generate a large force, which in turn can move a larger and heavier diaphragm. The loudspeaker can thus be used to generate very intense sound waves.

The designer of a loudspeaker faces problems similar to those met in the design of the moving-coil microphone, notably the avoidance of mechanical resonance in the diaphragm. In addition, it is difficult to make a large diaphragm vibrate at high frequencies. The better-class loudspeakers get over this by fitting two diaphragms as in Fig. 12.12 (*a*); a very small diaphragm reproduces the higher frequency waves and is

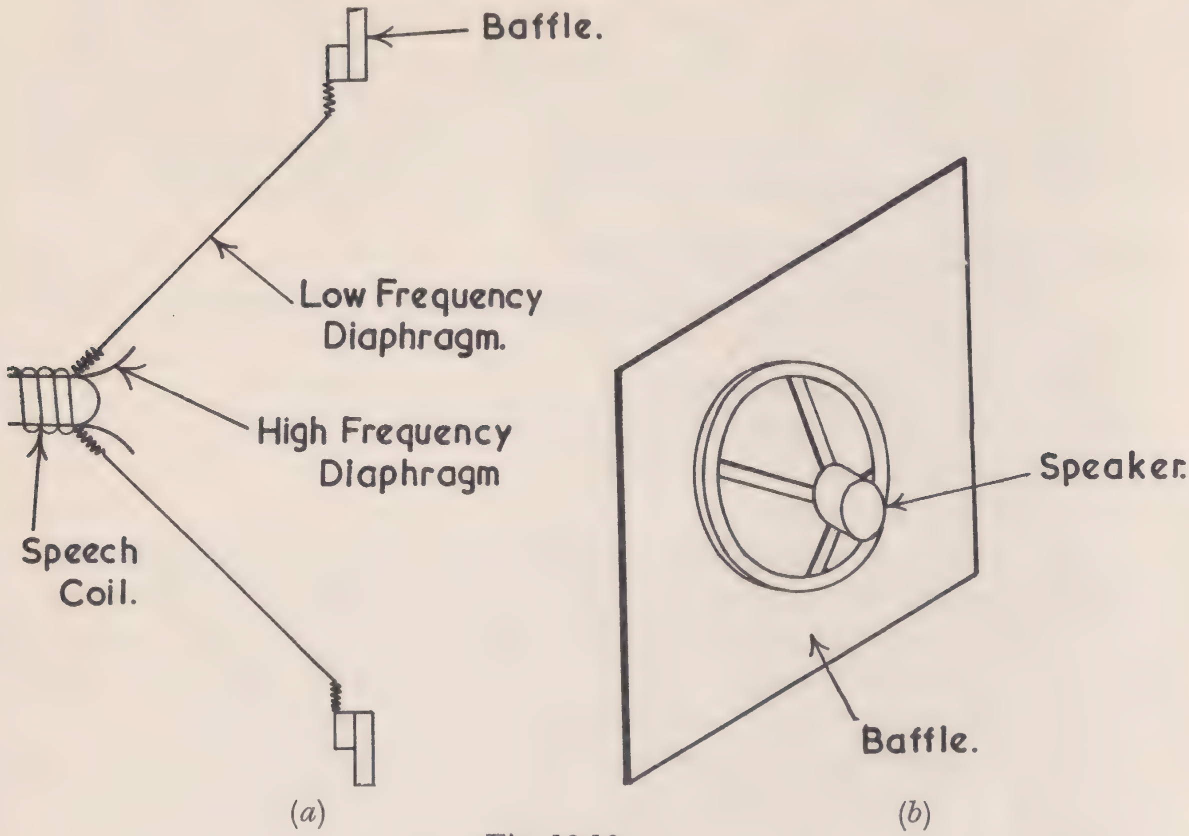


Fig. 12.12

coupled through a slightly flexible collar to a larger diaphragm, which receives only the lower frequency components.

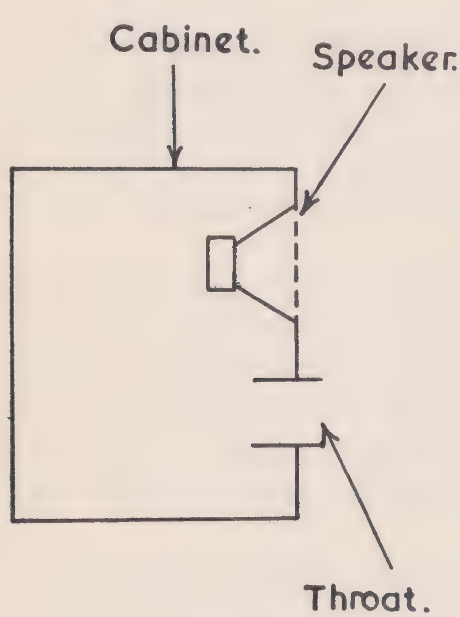


Fig 12.13

A loudspeaker is normally mounted on a large size baffle-board as in Fig. 12.12 (*b*). The front of the cone emits compression waves at the same time as the rear of the cone generates rarefactions; these two must not be allowed to unite or they would cancel each other out. An alternative mounting is shown in Fig. 12.13, the cabinet forms a cavity or Helmholtz resonator, tuned by the throat to resonate at a very low frequency, such as 20 cps; this improves the bass response of the speaker.

(c) The Telephone Receiver or Headphones

This piece of apparatus is shown in Fig. 12.14. Basically it consists of two soft-iron cores on which coils consisting of many turns of fine wire are wound. If a current is passed through these coils, the cores

become magnetic and attract a metal diaphragm; this is usually made of Stalloy, which is an alloy having good magnetic properties for this purpose. However, it will be noticed that the cores, if magnetised, will

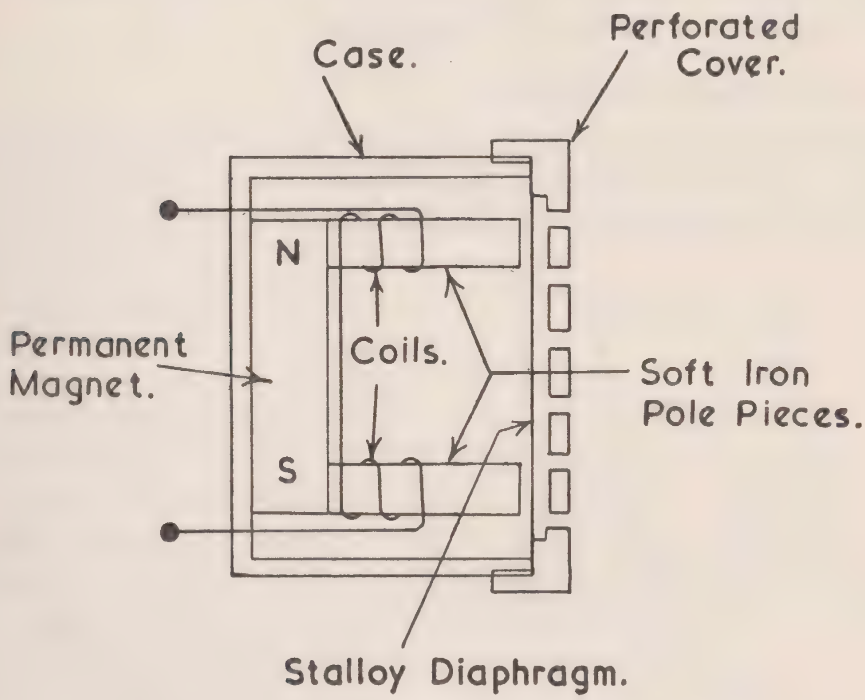


Fig. 12.14

attract the diaphragm whatever their polarity, resulting in a doubling of frequency. This effect is avoided by introducing a permanent magnet (as shown in Fig. 12.14) which induces poles in the soft-iron cores and results in the diaphragm taking up a bowed-inward shape. The coils are wound so that current flowing one way in them strengthens the induced poles and so the diaphragm is attracted still further inwards, whilst if the current reverses in direction it weakens the poles and allows the diaphragm to spring back to its flat condition. Thus positive or negative cycles of input current cause respectively inward or outward motion of the diaphragm and frequency doubling is avoided.

The telephone receiver is meant to be pressed closely against the ear and consequently is not designed to produce sound waves of great intensity. It is, however, of great sensitivity, a minute electrical current being capable of giving an audible sound output. The diaphragm is fairly light and quite stiff, thus its mechanical resonance frequency is rather high, and its efficiency below this frequency is rather poor, i.e. it does not reproduce the lower notes very well. This, however, is not a serious drawback, for the receiver was designed for the reproduction of speech and it is found that the intelligibility of telephonic speech is increased if the intensity of all frequencies below about 200 cps is reduced.

The telephone receiver does not give a completely faithful reproduction of the electrical waveform supplied to it; the diaphragm has several modes of vibration (see vibration of flat plate, page 34) and this results in the generation of spurious frequencies in the output, some of which are not harmonics of the input frequencies.

(d) The Gramophone Record

The gramophone record, as is well known, consists of a long spiral groove cut in the surface of a disc of hard plastic. If the grooves are seen under a microscope, it will be noticed that they have a wavy form.

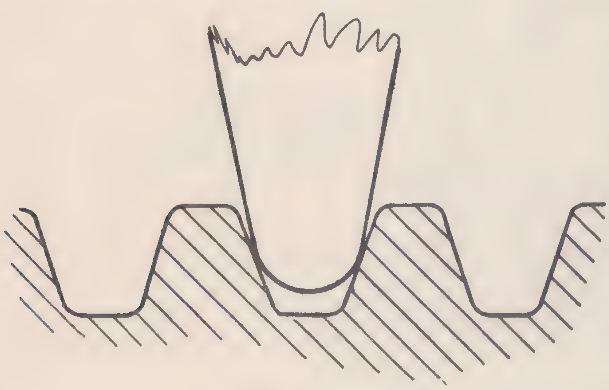


Fig. 12.15

When a record is 'played back' it is rotated at a constant speed on a turntable and some sort of 'pick-up' is used to hold a needle against the record, the needle fitting into the groove as shown in Fig. 12.15. The needle mounting in the pick-up is arranged so that the needle can move freely for a small distance in a radial direction across the record but is held stiffly in other directions; thus as the record rotates, the wavy grooves move the needle from side to side. This motion of the needle

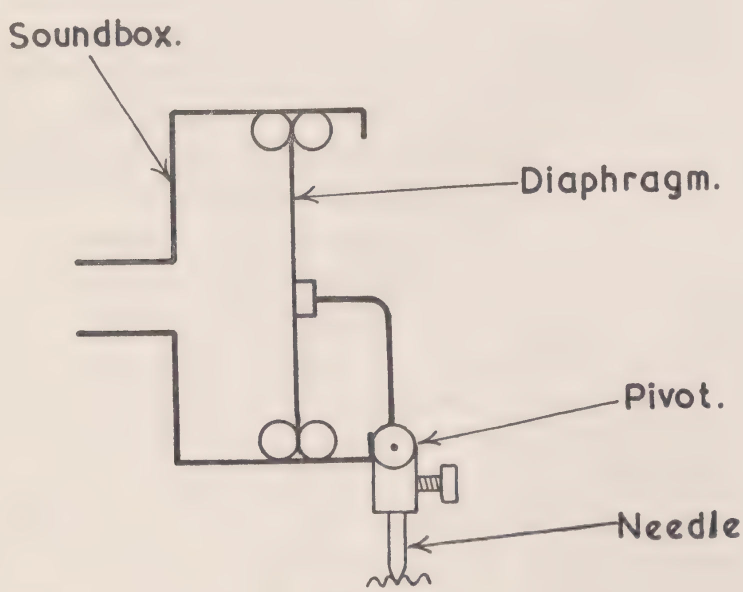
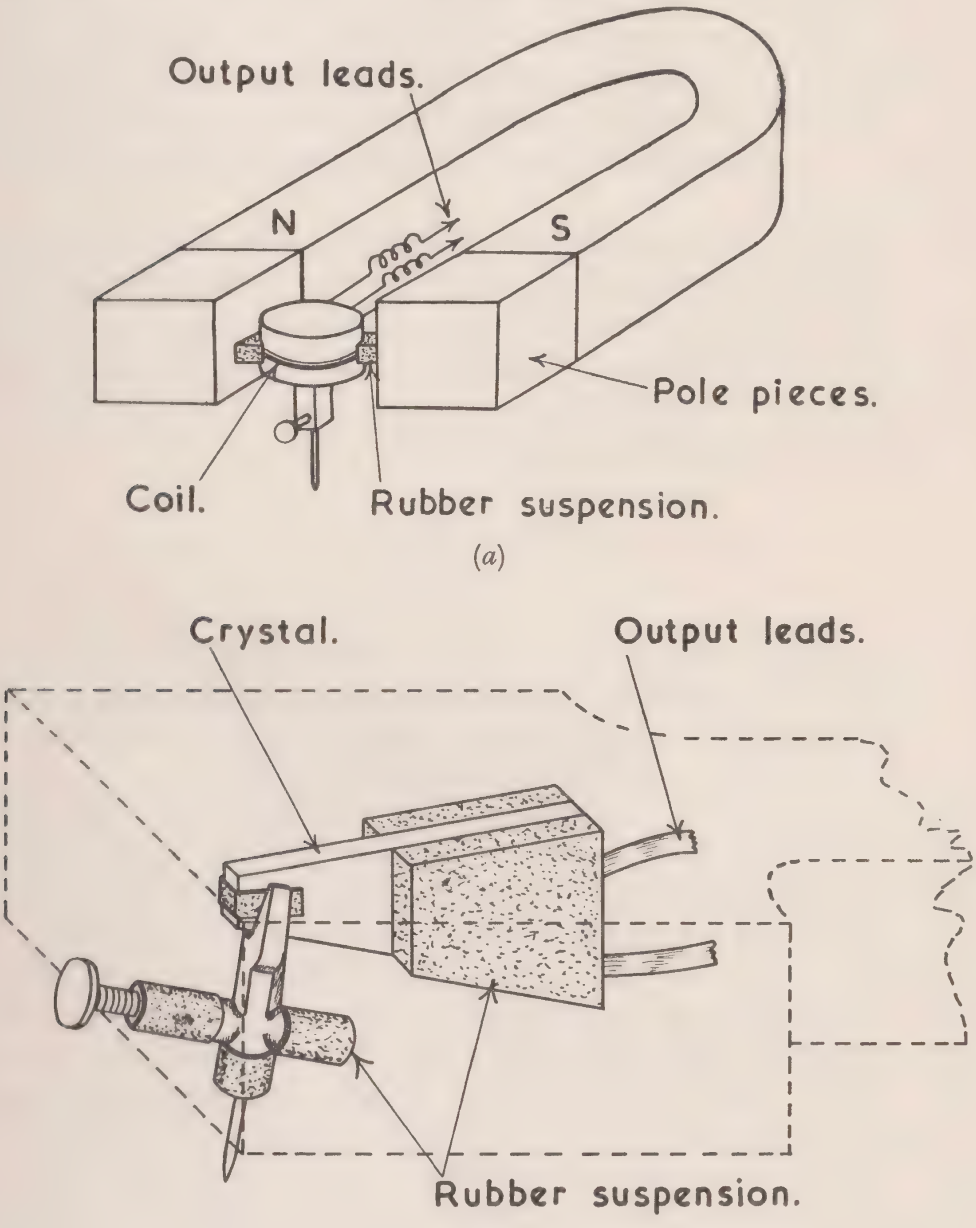


Fig. 12.16

can be used to vibrate a diaphragm as in the 'soundbox' of the old-fashioned acoustic gramophone (Fig. 12.16); or to operate one of the more modern forms of pick-up.

The moving-coil pick-up is shown in Fig. 12.17 (*a*), motion of the

needle vibrates the coil in a magnetic field and so generates an alternating voltage. In the crystal pick-up (Fig. 12.17 (*b*)), the needle bends a crystal of Rochelle salt, this gives rise to an alternating voltage as described previously. The characteristics of these pick-ups are similar to those of the corresponding microphones.



(*a*)　(*b*)

Fig. 12.17

The record is made by traversing a 'cutting head' across a blank disc of wax as it rotates. The cutting head is very similar to a moving-coil pick-up, but is used in the opposite sense, that is, an alternating current is passed through the coil causing it to move backwards and forwards in the magnetic field; this oscillates the cutting stylus from side to side and so cuts the wavy spiral groove in the record. The stylus

is a diamond-tipped needle, the diamond being ground to the shape of groove required. The wax record is too soft for general use and so it is copied in metal by an electro-plating technique and this master is used to press new records in a hot plastic.

It can be shown that if a gramophone record is to give the same output at all frequencies, then the amplitude of the waves in the groove should be inversely proportional to the square of the frequency. At very low frequencies, the waves would become very large and might cut into adjacent grooves, to avoid this, the amplitude of the waves is kept constant at all frequencies below 250 cps; this introduces a loss of 6 db per octave.

The normal 78-rpm records were originally designed to be played with a steel needle having a tip of diameter about three thousandths of an inch; if the waves in the grooves become smaller than the needle tip, it will cease to follow them properly and the output will drop. At the inside edge of a record (radius about $2\frac{1}{2}$ in) the length of one wave of frequency f cycles per second is given by

$$l = \frac{2\pi}{f} \cdot \frac{5}{2} \cdot \frac{78}{60} \text{ in.},$$

and thus one loop of the wave becomes smaller than the needle tip at about 3·5 kcs. (1000 cps = 1 kilocycle per sec, abbreviated to 1 kcs). Hence the response of a gramophone record is roughly constant from 250 cps up to 3·5 kcs, but falls off outside this range.

Modern methods of manufacture have enabled us to produce sapphire needle tips with a radius of only 0·0005 in., i.e. one third of the size of the steel needle. These will either permit better reproduction of 78-rpm records or allow the record to be run slower for the same quality, and this has led to the production of long-playing records, representing a compromise between the two cases. These records rotate at either 45 or $33\frac{1}{3}$ rpm, i.e. about half the speed of normal records. Due to the smaller needle tip, narrower grooves can be used, giving about 250 to the inch against 100 to the inch on normal records, thus the playing time is increased about fivefold. Further, the quality is improved, for whilst the needle size is reduced by a factor of three, the speed is only halved. The better quality of reproduction obtained with long-playing records, especially the 45-rpm records, makes it necessary to use a different base material for the discs. 78-rpm records are made of a mixture of shellac and resin filled with graphite and powdered slate, the grains of slate cause a background hiss at about 5 kcs and upwards. This noise cannot be tolerated with the improved high-frequency response of long-playing records, and so they are made of vinylite, a plastic material which offers a very smooth surface.

(e) Sound Film

The basic principles of recording sound on film are as follows. The sound to be recorded is converted to an electrical waveform by means of a microphone and amplifier. This waveform is used to control the amount of light focused on a narrow strip at the edge of a photographic film, which runs past the sound-recording head at a steady rate; consequently, after development, a strip more opaque in some places than in others appears along the edge of the film.

To play back the film, a narrow beam of light is directed on to the sound-track part of the film; the light which is transmitted passes into a photo-electric cell and this in turn generates an electric current proportional to the intensity of light falling on it. The light entering the cell will have intensity variations similar to the waveform of the original sound and thus the current output of the cell will also represent the original sound. This current can be amplified and used to energise a loudspeaker.

The film in a ciné camera or projector moves through the gate in jerks, this, of course, would make it impossible to record sound at the same time. Instead, the film is run through the sound head after passing through the gate, and is driven over the sound head by a set of constant-speed sprockets. A slack loop of film is left between the gate and the sound head to absorb the jerking motion.

(f) Tape Recorder

The tape recorder is becoming a popular instrument for 'home' recording at the moment. The basic idea is not new, however, for patents for recording on steel wire or tape were taken out in the last century.

The modern recorders use a tape made of a plastic impregnated with a very finely powdered magnetic material. The tape is drawn at a steady speed past a recording head as shown in Fig. 12.18. The head consists of a magnetic core *ABCD* having a very narrow gap (rather less than a

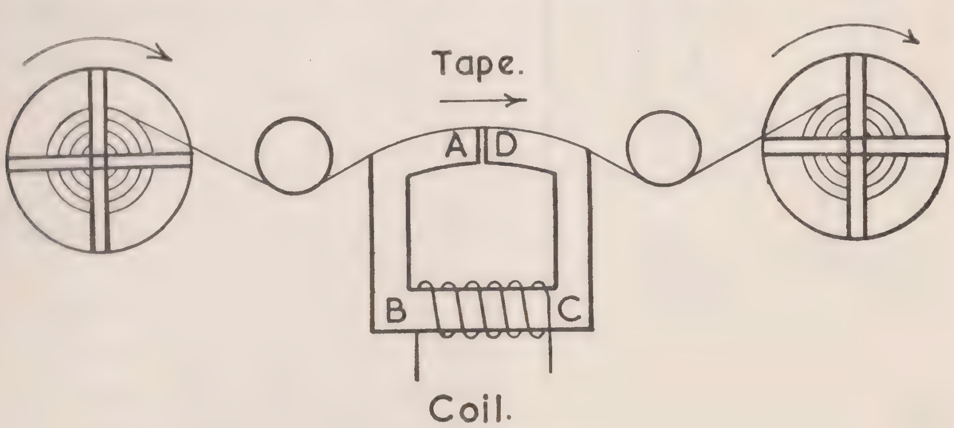


Fig. 12.18

thousandth of an inch wide) between the poles at A and D. If current is passed through the coil, the core will become magnetised; the lines of force, in preference to jumping the gap between the poles, will pass through the magnetic material of the tape bridging the gap; this will produce a very small patch of magnetism in the tape and if the input current is alternating, a succession of magnetic patches will appear along the tape.

To play the tape back, it may be run past a similar head, having the coil connected to an amplifier and loudspeaker. When a magnetic patch bridges the gap, lines of force will flow round the core, cut the windings of the coil and induce a voltage in it. This voltage is then amplified and applied to the loudspeaker.

The frequency response of a tape recorder depends to a certain extent on the speed at which the tape runs through the machine. At a speed of 15 in. per sec the response is rather better than the best gramophone records. This speed is used by some gramophone companies to make the original recording of a piece of music, which is then transcribed to gramophone records.

The machines available commercially, however, run at either $7\frac{1}{2}$ or $3\frac{3}{4}$ in. per second. Of these, the faster gives a response about the same as a good-quality gramophone record but at a speed of $3\frac{3}{4}$ in. per second, the response is noticeably inferior.

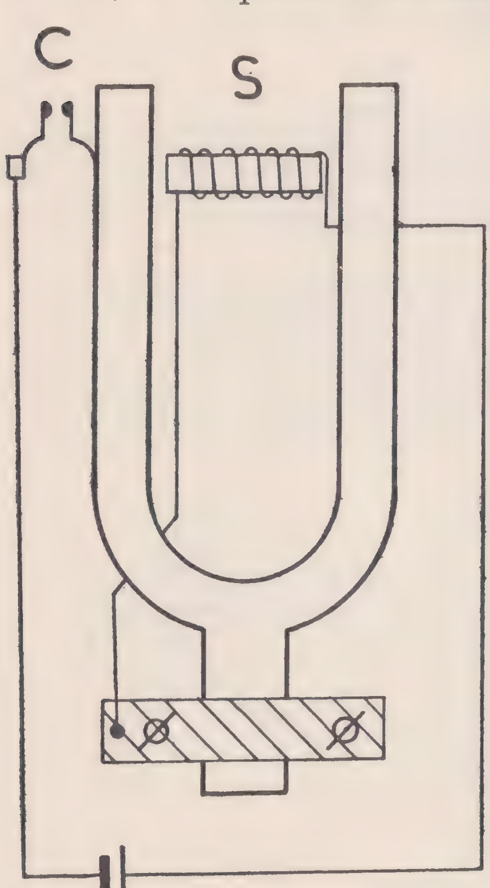


Fig. 12.19

The tape recorder is long playing and very convenient to use, for the record can be made and played back immediately without any intermediary processes. Also the instrument can be handled by anyone, whereas cutting a gramophone record is a job for a highly skilled technician.

12.5 Instruments Used for Acoustic Measurements

(a) Electrically Maintained Tuning-fork

The tuning-fork still remains the best method of producing a sound wave of accurately known frequency; standard forks are made at many frequencies within the range 50 to 5000 cps. The inconvenience of having to strike the fork (and the damage done to the fork thereby) are avoided by maintaining the

vibration of the fork electrically. Two methods of doing this are described below.

In the simpler method, a small iron-cored solenoid S is placed between the tines of the fork (Fig. 12.19), the whole instrument being mounted on a rigid baseboard. The core of the solenoid is arranged to be just clear of the tines when they are making their maximum inward vibration. One of the tines also carries a very small platinum contact C, mounted on a light flexible steel strip; a similar contact is mounted on the baseboard and is adjusted so that the contacts close when the tine begins to move outwards from its rest position.

The contacts and the winding of the solenoid are connected to a battery as shown in the diagram. If the fork is set vibrating by hand, the contacts close when the tines are flexed outwards and this allows a current to flow from the battery; the circuit is made through the contacts into the metal of the fork, thence through the solenoid and back to the battery. The current magnetises the core of the solenoid and attracts the tines of the fork inwards; this opens the contacts, cuts off the solenoid current and demagnetises the core. By this method the tines get an inward pull every time they are flexed outwards; by altering the current through the solenoid the strength of the pull may be made just large enough to keep the fork in steady vibration.

This method of maintaining a fork is open to some objections—the tines should perform a simple harmonic motion, but the impulse of each cycle distorts the motion slightly and introduces harmonics into the note produced by the fork. The frequency of the fork also depends to some extent on the size of the impulse—that is on the current flowing through the solenoid. It so happens that these effects are minimised if the fork is very large and of low frequency—usually about 50 to 100

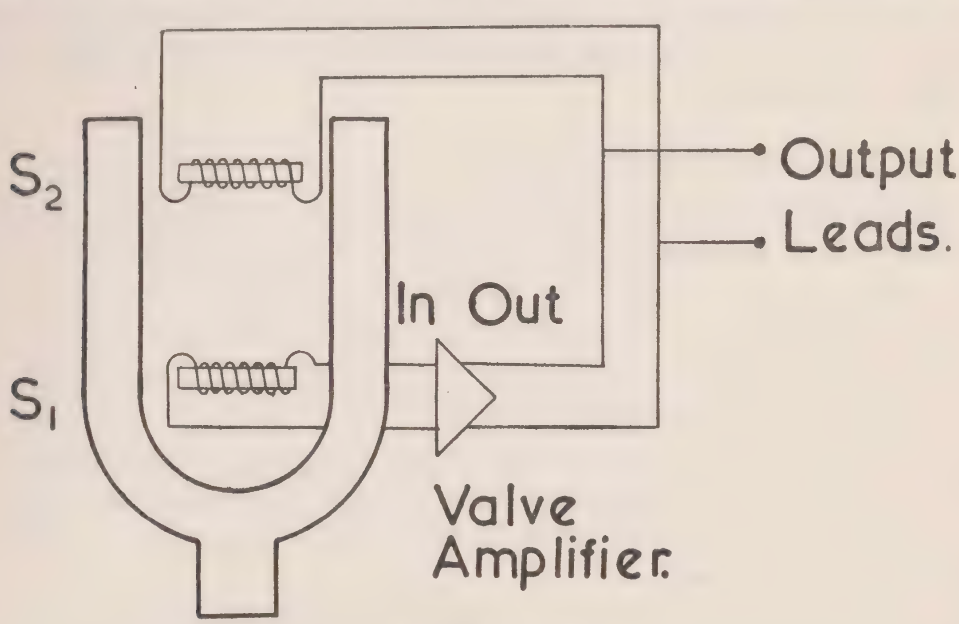


Fig. 12.20

cps—and this method of maintaining a fork is generally restricted to frequencies in this range.

The second method is illustrated in Fig. 12.20; two solenoids are placed between the tines, the one nearer the base of the fork being connected to the input of a valve amplifier, whilst the other solenoid is connected to the output. The core of S_1 is a permanent magnet, whilst that of S_2 is soft iron.

If the tines of the fork are set vibrating, the air gap between the core of S_1 and the tines is alternately increased and decreased; this causes a variation in the magnetic flux through the core which in turn induces a voltage in the winding of the solenoid. With careful design and positioning of the coils, the waveform of the induced voltage can be made to follow very closely the displacement curve of the tines.

The voltage is applied to the input of a valve amplifier and the magnified output voltage used to energise S_2 so that the magnetism induced in its core assists the motion of the tines.

By this means the vibration of the fork receives gentle assistance throughout the whole of the cycle, in contrast to a kick at the beginning of each cycle. If the gain of the amplifier is so adjusted that the assistance from S_2 just makes up for the natural damping of the fork, then it will continue in vibration indefinitely. The note produced in this fashion is very pure and contains practically no harmonics, nor is the frequency dependent upon the conditions in the maintaining circuit. Further, some of the output of the amplifier can be led off to a loudspeaker, and this enables the fork to be supported on anti-vibration mountings in a thermostatically controlled oven, all of which contributes to the frequency stability of the system.

A typical fork, made of Elinvar (a steel alloy whose elastic properties are practically independent of temperature) and designed to vibrate at 1000 cps, can be expected to maintain that frequency within 0·0001 per cent. indefinitely.

(b) Beat Frequency Oscillator

Experiments in acoustics are often facilitated by the use of a source of sound whose frequency can be varied easily between fairly wide limits—from 10 cps to 10 kcs is a normal requirement.

Valve oscillators can be made to operate at audio-frequencies, but unfortunately variable tuning over a wide range of frequencies presents a problem. The variation in frequency of an oscillator is usually provided by a change in size of one of the electrical components in the circuit (normally a variable condenser); such variable components are not available commercially with a range large enough to cover the frequency scale quoted above.

The solution to this problem is found in the use of a Beat Frequency Oscillator, usually called a B.F.O. This makes use of the fact that if

two notes very close in pitch are sounded together, then a beat note is heard at a frequency equal to the difference between the two original frequencies. The beat note can be heard even if the combining sounds are just above the audible range of the ears. Similarly, if two high-frequency electrical waveforms are generated by valve oscillators and then added together, a waveform appears at the difference frequency and can be applied to a loudspeaker and so give rise to sound waves. A typical circuit arrangement of a B.F.O. is shown in Fig. 12.21; of the two oscillators, one is fixed at a frequency of 100 kcs while the other can be tuned from 90 to 100 kcs; the output of each of these two

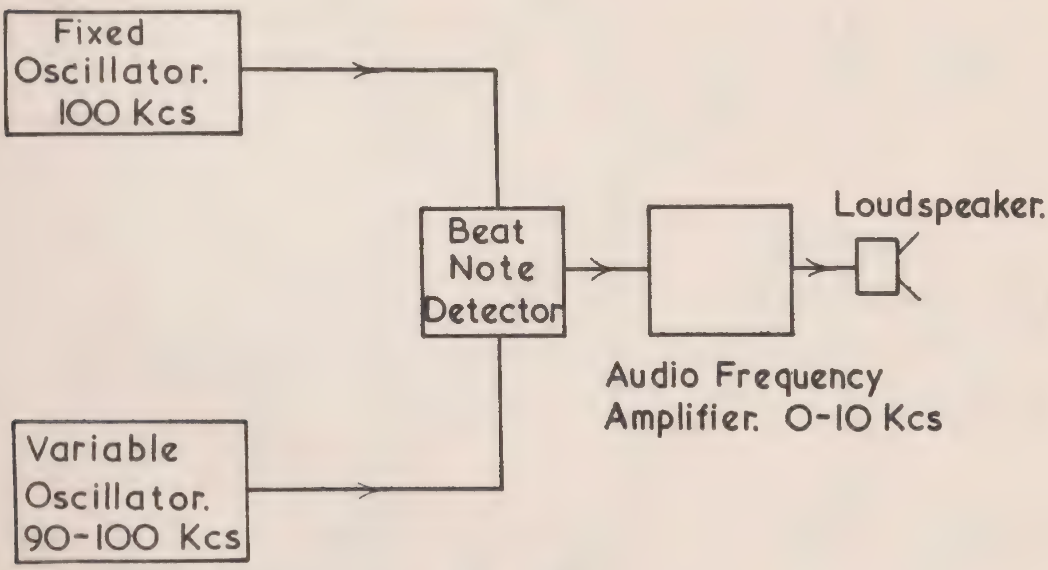


Fig. 12.21

oscillators is applied to a detector circuit, which mixes the two high-frequency waveforms and gives a waveform at the difference frequency as its output. It will be seen that as the second high frequency oscillator is tuned from 90 to 100 kcs, the beat note drops from 10 kcs down to zero frequency. The waveform at audio-frequency is further amplified and then used to energise a loudspeaker and so converted into sound waves.

(c) Frequency Meters

The methods of measuring frequency described in previous chapters are not of high accuracy, neither are they suitable for industrial use; to overcome this, various meters have been devised to measure frequency quickly and easily. The simplest of these employs a range of metal strips, called reeds; each one is cut to a different length and vibrates naturally at a specific frequency. The note whose frequency is to be measured is picked up by a microphone and converted into electrical impulses occurring at the same frequency as the pressure pulses in the original sound wave. These pulses are used to energise an electro-

magnet which attracts small pieces of soft iron mounted on each of the reeds. As a result, the reed whose natural frequency of vibration coincides with that of the sound builds up to a large oscillation, while all the others execute only a very slight vibration.

A second type of frequency meter picks up a note in a microphone and converts the sound waveform into a similar electrical waveform.

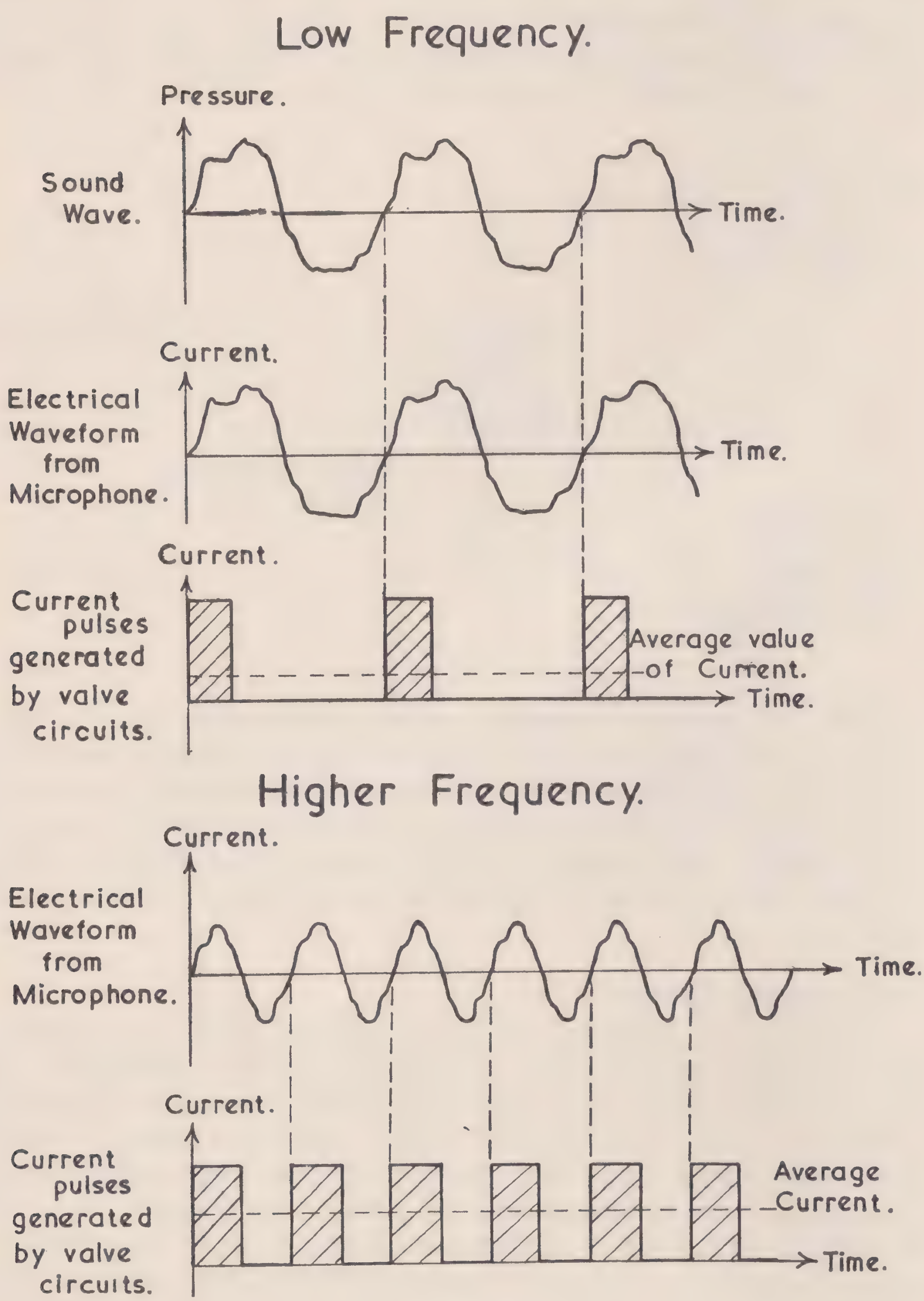


Fig. 12.22

Various electrical circuits then generate a square-shaped pulse of current of fixed amplitude and duration every time the current in the original waveform changes from negative to positive (see Fig. 12.22).

These pulses are passed through a current meter, which measures the average value of the current. Obviously the higher the frequency of the original sound, the closer together are the pulses of current and the higher its average value. Thus the meter reading is related to the frequency and can be calibrated in terms of frequency instead of current.

The accuracy of both of the meters described above depends upon the care with which the manufacturer carries out the calibration of the instrument.

There is, however, another type of meter which does not rely on calibration. This uses electronic valves arranged so that they can count electrical pulses and show the result of such a count on a meter. (Valve circuits of this type form the basis of electronic computers or 'brains'.) The incoming sound wave is converted to current pulses as in the previous case and these pulses are counted by a valve counting circuit which is switched on for exactly one second by a standard pendulum beating seconds. The number of pulses which the circuit counts in this time is the frequency of the source.

(d) Cathode-ray Oscilloscope

The cathode-ray oscilloscope, developed during the last twenty-five years, has proved of inestimable use in the study of acoustics, for it enables the waveform of any sound to be displayed on a screen for visual examination—some effects, such as the presence of an unwanted second harmonic, are more readily detected in this fashion than by ear.

The instrument is built around the cathode-ray tube, similar to that used in a television set, and shown diagrammatically in Fig. 12.23.

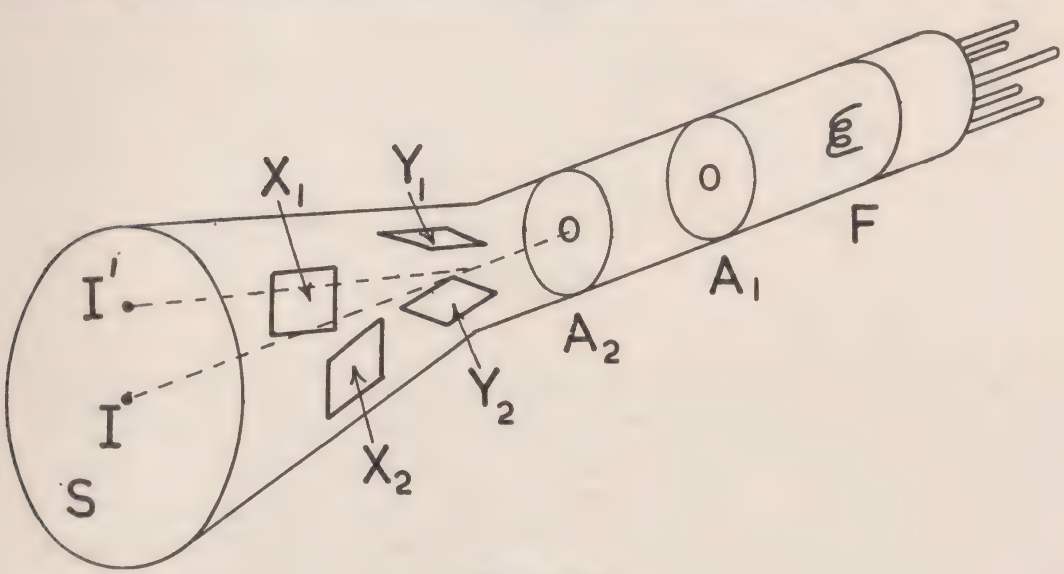


Fig. 12.23

A coil of wire F, called the filament, is heated by an electric current and gives off a copious supply of electrons; these are attracted very strongly by two electrodes A_1 and A_2 which have very high positive potentials applied to them, A_2 being higher than A_1. These electrodes each have a small hole pierced at the centre, consequently some electrons shoot right through the pair of holes and emerge as a narrow beam of electrons travelling towards the screen S; the inside of the tube is evacuated so that the path of the electrons is not impeded by collisions with air molecules. This electrode assembly is usually called the 'electron gun' as its purpose is to shoot a beam of electrons at the screen. Modern tubes have a more refined gun, details of which will be found in books on electricity. The screen is coated with a material which fluoresces when bombarded with electrons, consequently, when the gun is operating, a small bright spot, I, should be seen at the centre of the screen.

On emerging from the gun the electron beam passes between two more pairs of plates, Y_1 Y_2 and X_1 X_2, these are called the deflector plates; if a positive potential is applied to Y_1 and negative to Y_2, the electrons will be attracted upwards as they pass between these plates, thus the whole beam will be deflected upwards to strike the screen at I'. Similarly, a potential difference applied between X_1 and X_2 will deflect the spot sideways. If the screen is imagined to be squared off in cartesian co-ordinates, then it will be seen that the plates X_1X_2 deflect the spot along the X-axis while the Y_1Y_2 plates cause motion of the spot along the Y-axis.

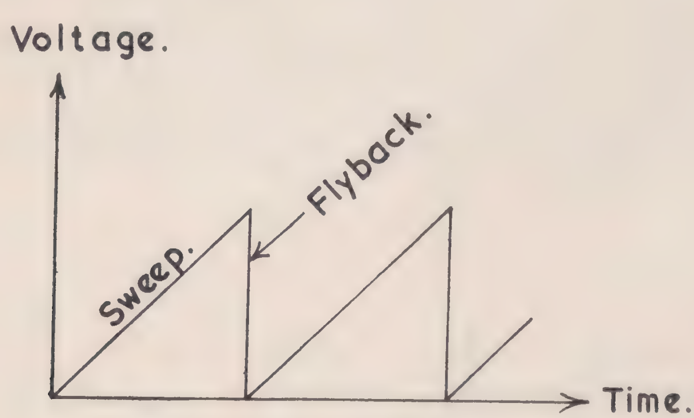


Fig. 12.24

The cathode-ray tube is normally mounted in a chassis which contains the power supplies for the filament and anodes, also a piece of electrical equipment called a *time base*. This is an electrical circuit designed to generate a voltage which grows at a constant rate from zero to some predetermined value and then drops back instantaneously to zero; a graph of the voltage is shown in Fig. 12.24.

If this voltage is applied to the X plates, it will sweep the spot steadily in a horizontal straight line across the screen and then, when the voltage drops to zero, the spot will fly back to its original position and start a second sweep. The position of the spot on the screen can thus

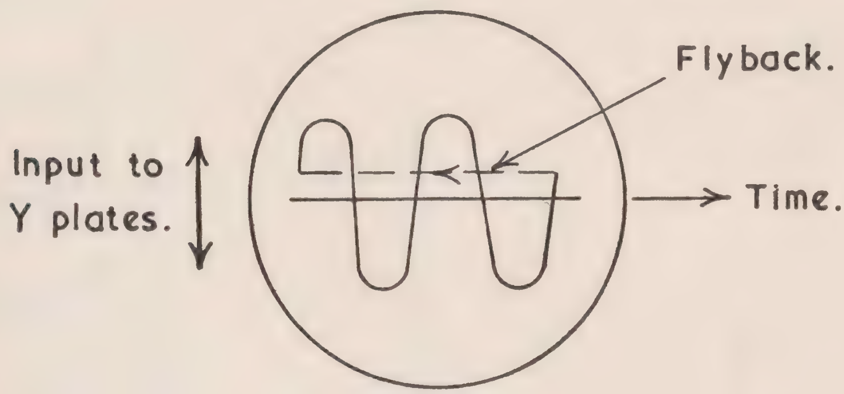


Fig. 12.25

be made to move horizontally to the right, the distance traversed increasing linearly with time; hence if the spot is deflected along the Y-axis at the same time, it will draw a graph on the tube face of the input waveform to the Y plates against time along the X-axis as shown in Fig. 12.25. Persistence of vision and a short afterglow in the fluorescent material of the screen enables the whole of the trace to be seen at once.

The time base also has an electrical circuit, called the synchronising circuit; this ensures that the flyback occurs after a fixed number of cycles of the input waveform if it is periodic and that subsequent traces are drawn on the screen in the same position as the previous one.

It is this complete instrument consisting of cathode-ray tube, power pack and synchronised time base which is called a cathode-ray oscilloscope, often abbreviated to C.R.O.

The uses of the C.R.O. will now be apparent, for if a sound is converted to an electrical waveform by any of the means described previously, this waveform can be applied to the Y plates and the waveform is immediately available on the screen for visual examination.

(e) Lissajous' Figures displayed by a C.R.O.

In this section we are going to consider the motion of a system which is subjected to two separate simple harmonic motions, each of which produces motion in a different direction. For example the bob of the pendulum of a clock makes a motion which might be illustrated by the line AB, Fig. 12.26 (a), whereas the motion of a ship in a gentle swell might be represented by the line CD (Fig. 12.26 (b)). If the clock is in the ship, then the motion of the bob in space is a combination of these two motions and is shown in Fig. 12.26 (c).

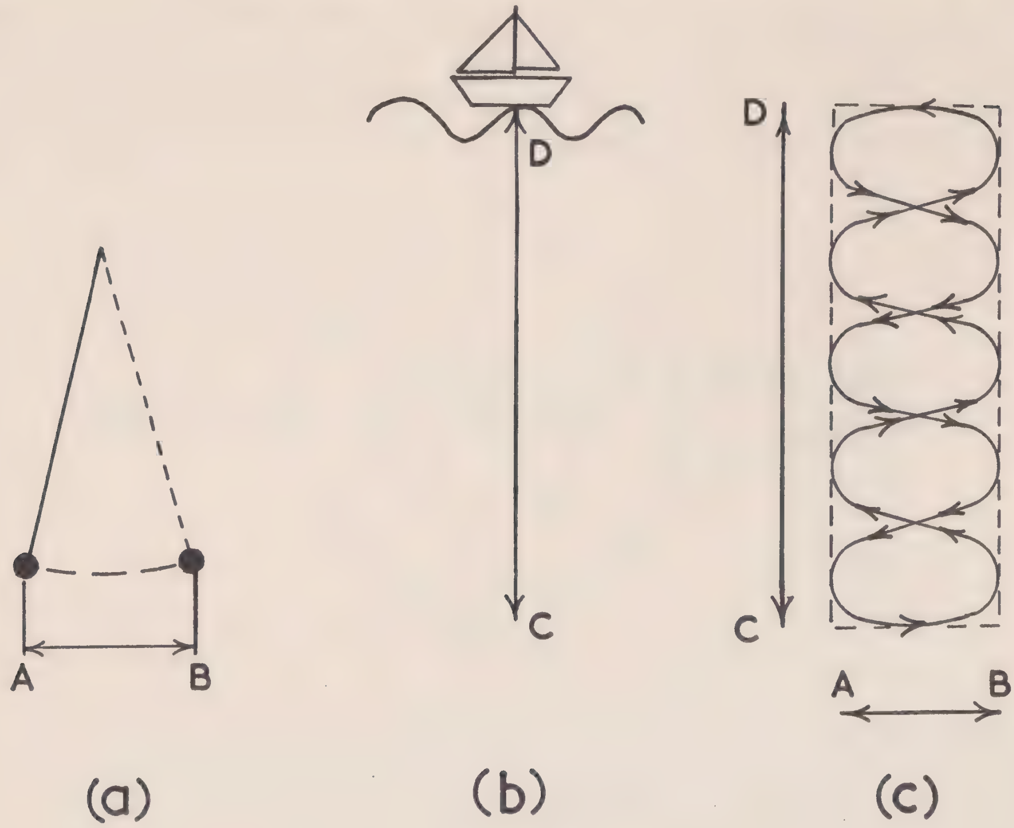


Fig. 12.26

This combination of two simple harmonic motions in different directions was studied by Lissajous and the patterns produced are called Lissajous' Figures.

In the case considered above, the two motions were at right angles and for simplicity the cases considered here will be restricted to this class.

The pattern to be expected from the combination of any two simple harmonic motions may be found by the graphical method described below. This process can be applied only if the frequencies of the two motions are in a simple ratio, i.e. 2:1, 3:1, etc.

It was seen in Chapter 5 that a simple harmonic motion could be produced by projection from a rotating vector; this method is used to produce the pattern illustrated in Fig. 12.27. Two vectors AA' and BB', of equal length (i.e. simple harmonic motions of equal amplitude) are projected on to the two lines OX, OY, at right angles and the two displacements combined as in a cartesian co-ordinate system. The two vectors start in phase (in position 1 they both have zero projections on their respective lines OX, OY), but one is moved on $22\frac{1}{2}°$ in equal increments of time (from position 1 to 2, 3, etc.) while the other moves on 45°. The lower vector thus completes a revolution in half the time taken by the one on the left and hence represents a simple harmonic motion of twice the frequency. We are thus combining two motions of

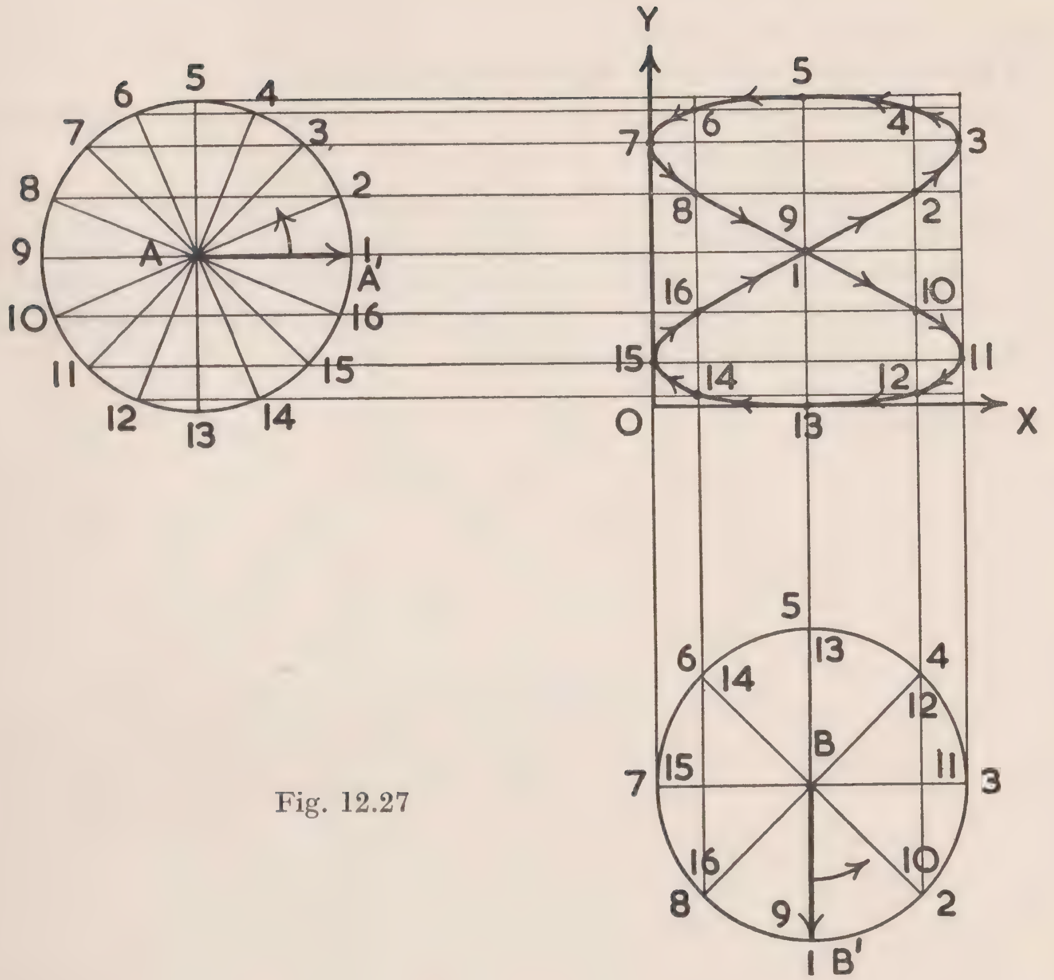


Fig. 12.27

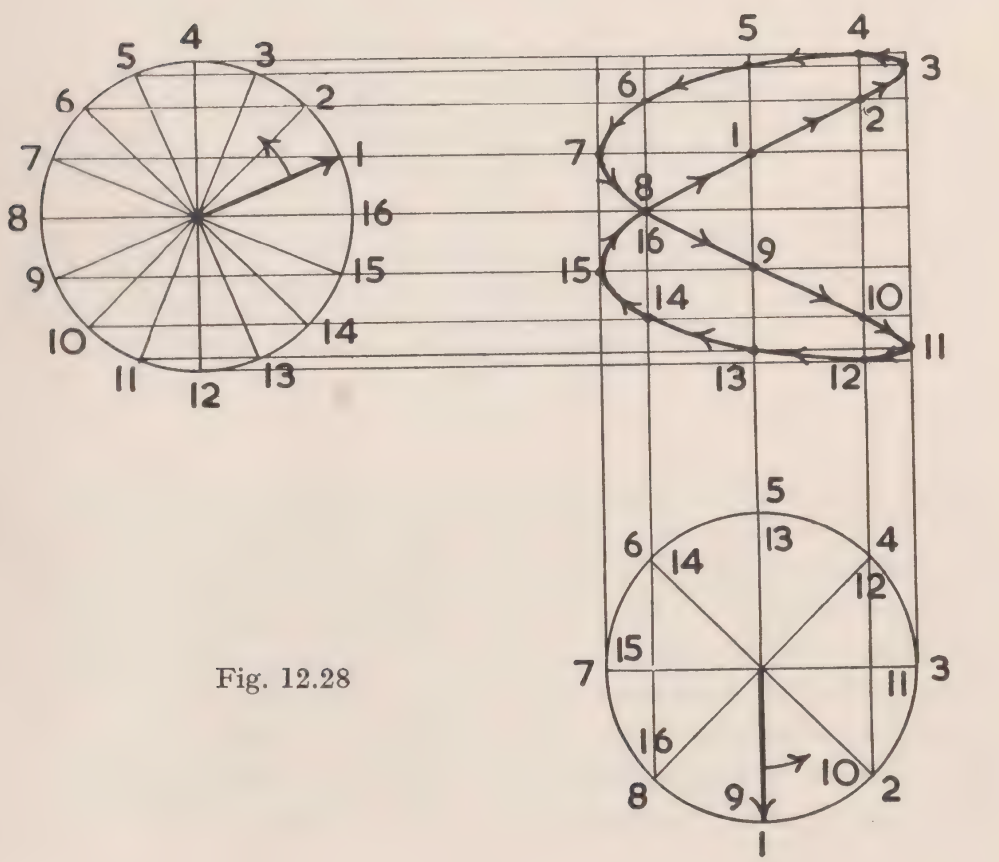


Fig. 12.28

equal amplitude but of frequencies in the ratio 2:1, starting in phase, and acting in perpendicular directions to each other.

The resultant pattern is a distorted figure-of-eight as shown, but has symmetry about two axes.

If the phase is different at the start of the motion (as shown in Fig. 12.28 where the vector on the left has been advanced by $22\frac{1}{2}°$) the figure is distorted somewhat and loses its symmetry about one axis.

Similar changes take place when the frequency ratio is altered and typical cases are shown in Fig. 12.29.

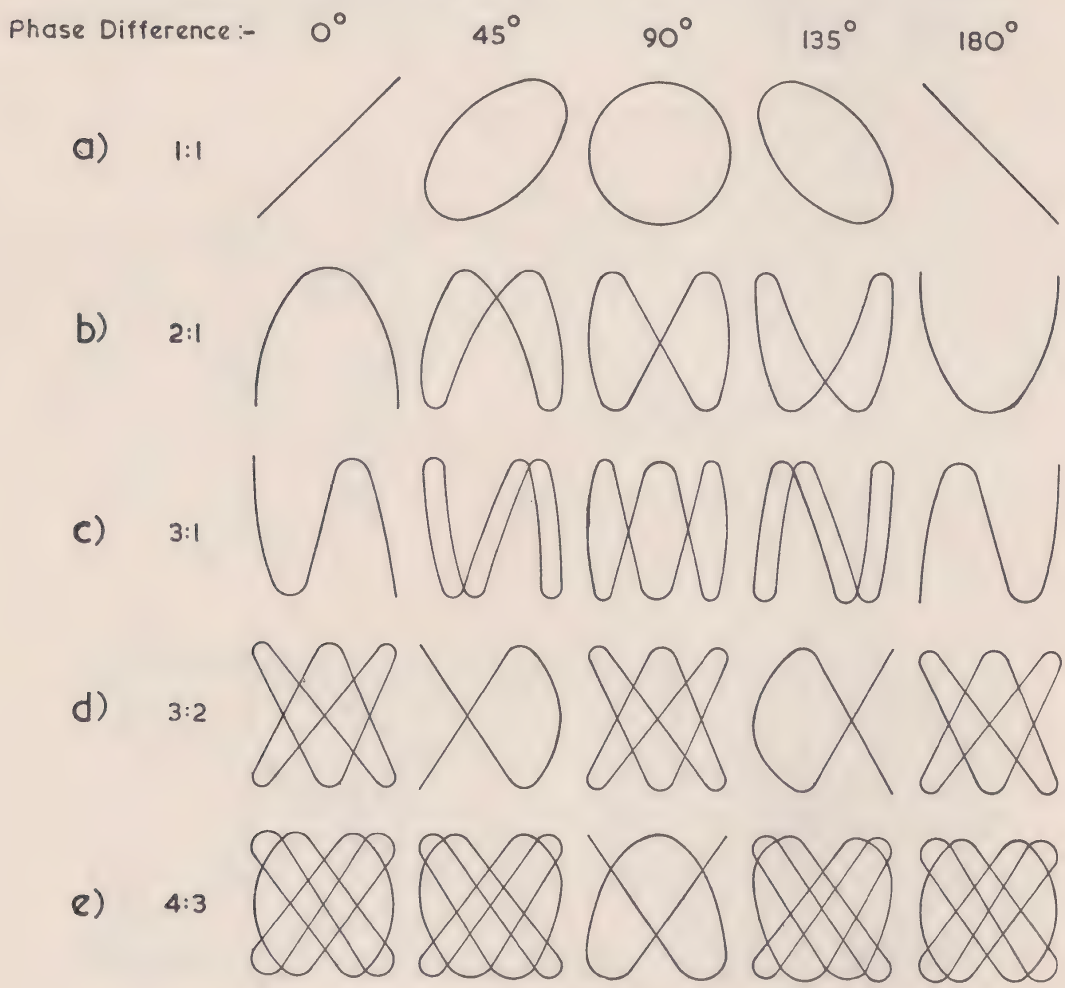


Fig. 12.29

Apart from certain degenerate figures, such as the extreme left of row (*a*), it will be seen that the ratio of the number of times the figure touches the top edge of the frame to the number of times it touches the right-hand edge is the same as the frequency ratio. Thus this method could be used to compare an unknown frequency with a standard frequency provided that one is a simple multiple of the other. The simplest way of displaying Lissajous' figures for this purpose is to use a C.R.O. The time base is switched off and one simple harmonic motion applied (as an electrical waveform) to the *X* plates while the other goes

to the Y plates, the motion of the spot of light then follows the pattern of one of Lissajous' figures.

If the two frequencies are not exact multiples of each other they can still be compared as follows: suppose that the frequency of an oscillator, nominally 1000 cps, is to be compared with the output of a standard 1000-cps fork. The output of each is applied to a C.R.O. and a Lissajous' pattern formed. The two sources are nearly in a frequency ratio 1:1, hence if they start in phase, the pattern will be a straight line as the left-hand figure of row (*a*), Fig. 12.29. Since they are not

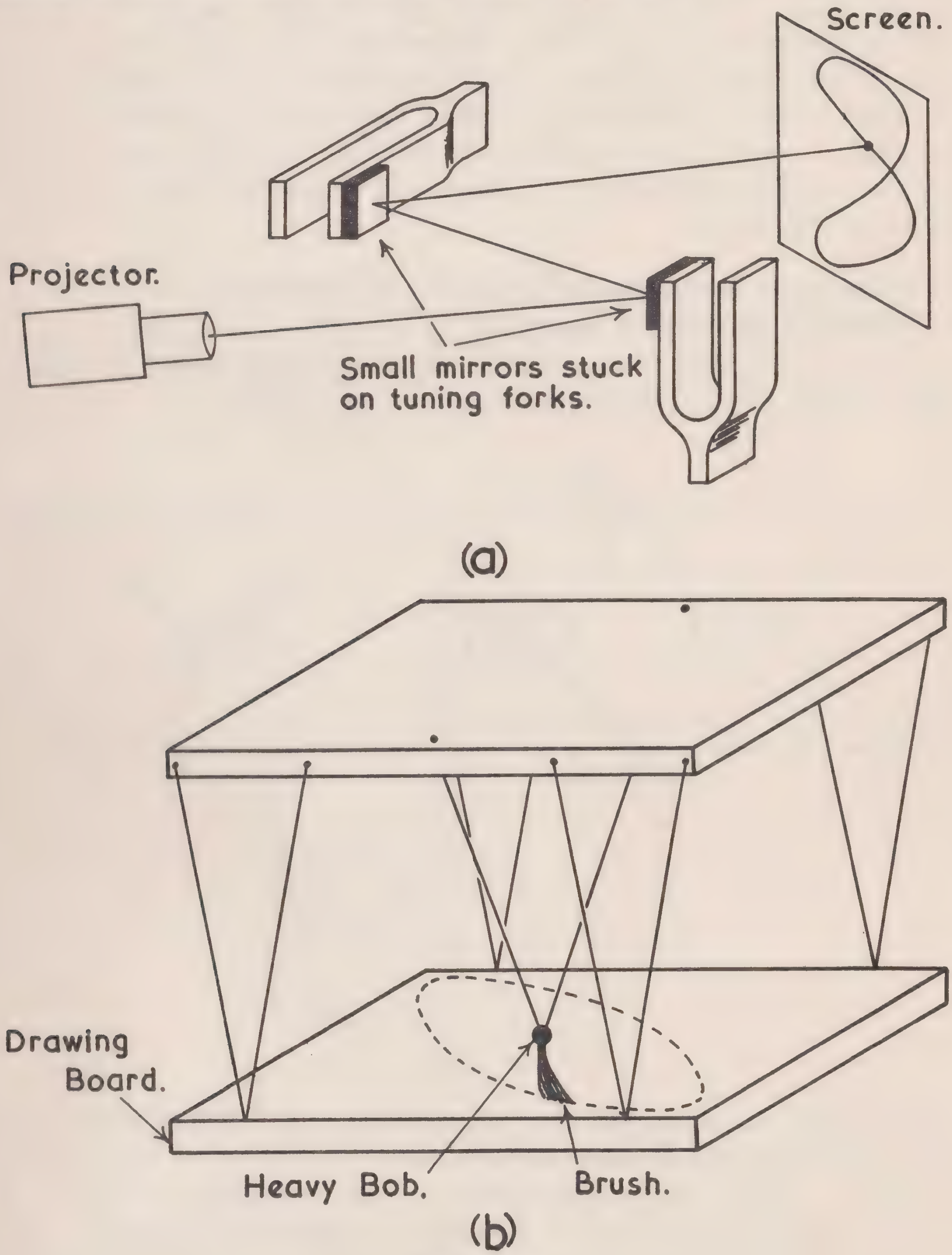


Fig. 12.30

exactly of the same frequency, one will gradually creep ahead of the other in phase, hence the straight line will open out into an ellipse, turning to a circle when the phase difference is 90°. This will be followed by an ellipse sloping the other way which finally turns into a line when the two sources are in antiphase. This cycle is then repeated the other way round as the two sources gradually creep back into phase. One source has now gained a whole cycle on the other, so that if the time taken for this to occur is measured, the difference in frequency can be calculated. In the experiment mentioned above, the pattern might change from a straight line through one complete range of patterns back to the same straight line in 50 seconds, thus one source gains 1 cycle in 50 seconds or 0·02 cycle in 1 second. The frequency of the oscillator is thus 1000·02 cps or 999·98 cps—to decide which of these is correct, a method such as that described on page 304 must be used.

Other mechanical methods of producing Lissajous' figures are illustrated in Fig. 12.30.

(f) Velocity of Sound using a C.R.O. as Indicator (Variation of Hebb's Method).

A famous method for measuring the velocity of sound is due to Hebb. This is shown diagrammatically in Fig. 12.31.

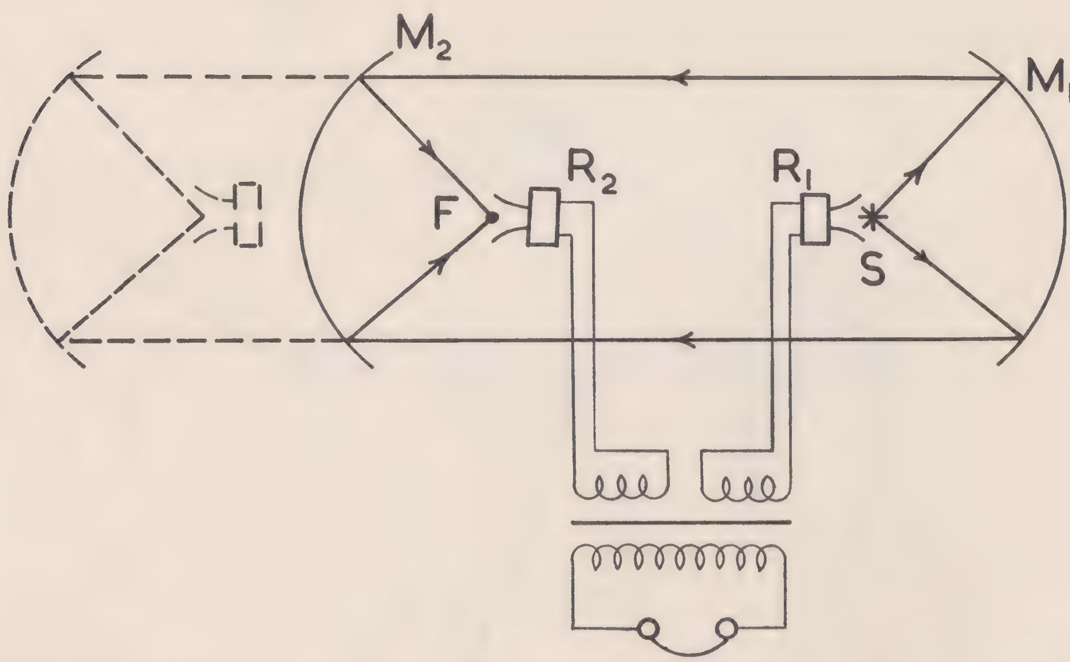


Fig. 12.31

Two large sound mirrors M_1, M_2 are used; the original ones were paraboloids five feet in diameter and made of plaster of Paris which provides a surface quite satisfactory for the reflection of sound. A source of sound S is placed at the focus of one mirror (a small loudspeaker energised by an accurately-calibrated B.F.O. might be used for

this purpose). After reflection, the sound should emerge from the mirror as a parallel beam which can be brought to a focus at F by the second mirror. If the beam is truly parallel, all the sound energy reflected from M_1 (apart from that absorbed by the air), will be collected and focused by M_2 and hence the intensity of sound at the focus F will not decrease very much as M_2 is moved away from M_1. Unfortunately it is impracticable to use a frequency much higher than about 2 kcs; this frequency has a wavelength in air of 6 in., hence the mirrors are only about 10 wavelengths wide and considerable diffraction effects occur causing spreading of the beam; this results in the intensity at F falling off as M_2 is moved away from M_1.

Two microphones R_1 and R_2 are placed so that one is near the source S and the other at the focus F. The output from each of these microphones was connected in Hebb's original method to two identical primary windings of a transformer. The sounds picked up by the two microphones may not be in the same phase. R_1 being close to the source, picks up the sound very nearly in its original phase; but the distance travelled by the sound from S to F via the two mirrors may be many wavelengths, thus the sound picked up by R_2 may be many cycles behind that received by R_1. If the input to each of the primary windings is in phase, the output of the transformer will be the sum of the two inputs, whereas if they are out of phase, the output will be the difference. If the source S, mirror M_1, and microphone R_1 are kept fixed while the mirror M_2, with microphone R_2 fixed at F, are moved slowly away from M_1, the path difference between the two mirrors will increase, and the phase at R_2 lag farther and farther behind that at R_1. The phase difference increases by one whole cycle for every wavelength increment in path length, thus the sound heard in the phones will rise and fall in intensity as shown in Fig. 12.32; the mirror moves one wavelength between each maximum of intensity. If a graph is

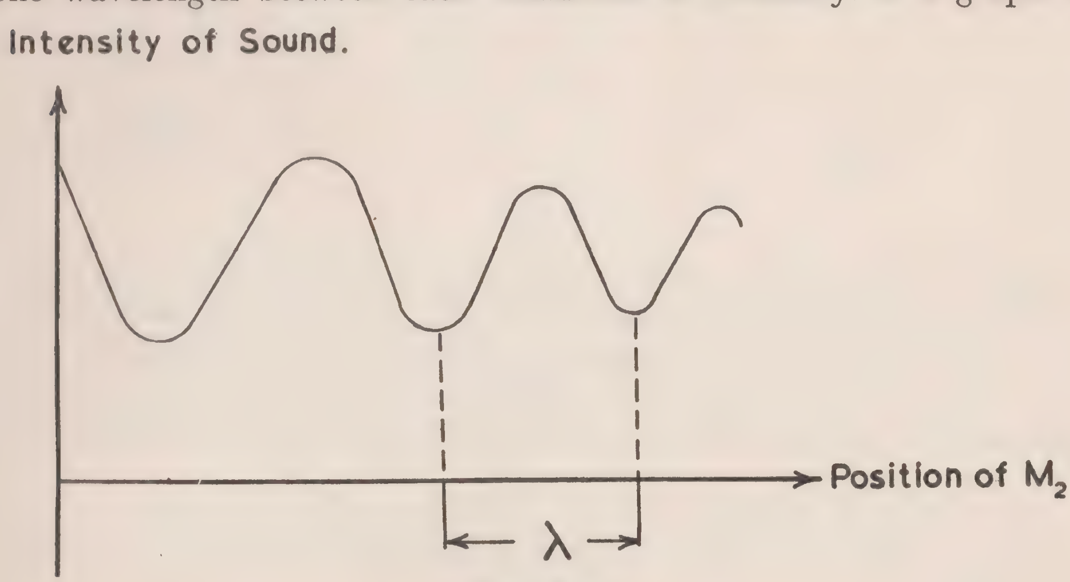


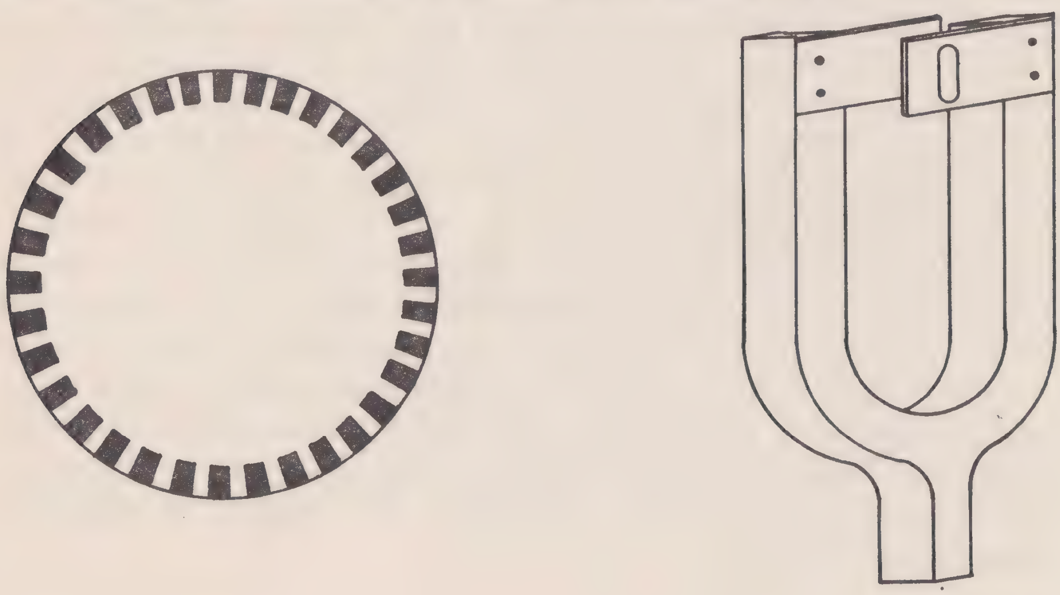
Fig. 12.32

plotted with the position of the mirror for each maximum as ordinates against the natural numbers as abscissa, then the slope of the graph gives a good estimate of the wavelength of the sound wave. The velocity of sound can be found from $v = f\lambda$, where f is the frequency of the source.

Headphones used in this way do not provide a very sensitive method of detecting a maximum; a much better method is to apply the outputs of the two microphones to the X and Y plates of a C.R.O. and so form a Lissajous' pattern. A position for M_2 can be found where the pattern is a straight line, and this will repeat itself wherever M_2 is moved by one whole wavelength, thus a graph can be constructed as described above.

(g) The Stroboscope

This instrument was originally developed as a means of measuring the frequency of mechanical vibrations or of rotational motion, but can also be used to measure the frequency of a sound wave.



(a) (b)

Fig. 12.33

Suppose that a disc, marked with black bars as in Fig. 12.33 (a), is spinning at a uniform rate of N revolutions per second, and that n black bars are drawn round the periphery. The angle between one bar and the next is $1/n$ of a complete revolution, but the disc completes one revolution in $1/N$ sec, thus it will turn through $1/n$ of a revolution in $1/nN$ sec.

Hence in this interval of time, the disc rotates so that one bar moves on until it occupies exactly the position vacated by the next bar along.

If the disc is illuminated by a light source which gives a very short flash every $1/nN$ sec, then at every flash the disc will be seen with the bars moved on one space; but since all the bars are the same, each picture will appear to be identical with the previous one and hence the disc will appear to be standing still.

Thus if f is the frequency of the flashing light, we may write

$$f = Nn,$$

hence f can be calculated if the rotational speed of the disc is measured. Alternatively, the rotational speed N can be calculated if f is known.

An alternative method of viewing a stroboscopic disc is shown in Fig. 12.33 (*b*). The tines of a very large fork carry plates, one of which has a slit cut in it. If the fork is set vibrating, this slit is uncovered for a brief instant when the tines are flexed outwards, thus the disc is seen through the slit once in every cycle of the motion of the fork. Sometimes both plates are slotted, the slots coinciding when the fork is at rest; this permits the disc to be seen twice in each cycle.

The result, however, is ambiguous to a certain extent, for if the disc were rotating at twice the speed given above, the bars would move on two spaces between each flash and the disc would still appear to stand still. On the other hand, if the disc is slowed down to half speed, the bars will move on half a space for each flash; thus once again the disc will appear stationary but the number of bars will seem to be doubled. The ambiguity can therefore be removed by arranging the speed of the disc to be the slowest which will give a stationary pattern without doubling the number of bars.

Instruments are available commercially to work in either fashion, i.e. either a light source flashing at a known frequency to determine the rotational speed of machinery, or discs rotating at known speed for frequency determination. In one model of this latter type, a microphone picks up a sound wave and converts it into electrical pulses, these are used to flash a light source at the same frequency as that of the sound. The disc is driven at a constant speed by a synchronous motor, and carries a number of concentric sets of black bars; the spacing between the bars in each circle is adjusted so that they correspond with the frequencies of the notes in a scale of equal temperament; the device can thus be used to tune keyed instruments such as a piano to an equal temperament scale.

12.6 Acoustics of Buildings

The acoustical properties of rooms and halls received little considera tion by architects until the beginning of this century; before that time buildings designed as theatres or concert halls were not planned so as to give the best possible reproduction of speech or music. The first systematic study of the acoustics of rooms was made by Sabine, an

American, when attempting to his own make lecture rooms suitable for speech.

He examined the factors which make a building acceptable for public speaking or the performance of music and he came to the conclusion, as others have done since, that by far the most important factor is the choice of a correct *reverberation* time for the buildings.

The walls, floors and other flat surfaces in any room reflect sound with only a small loss in energy; thus if a sound is generated in a room, the waves travel to the walls and are reflected back again—the wave may in fact suffer a hundred reflections or more before it becomes inaudible. A listener thus receives the sound by many paths, some much longer than others; thus a sudden impulsive noise may become drawn out into a long echo or *reverberation*.

Sabine defined the *reverberation time* of a room as the time taken by a sound to fall in intensity by a factor of a million (that is, to fall by 60 db), after cessation of the source.

If the reverberation time is long, it will cause confusion with later waves emitted by the source; speech loses its intelligibility and music becomes dissonant. On the other hand, too short a reverberation time makes the room sound 'dead'.

The best value for reverberation period depends on the use for which a building is designed, but, within broad limits, most people find that for speech about 0·5 sec is acceptable, whilst for music the reverberation time should be increased to one or two seconds.

The length of the reverberation depends on the reflecting properties of the walls, floor and ceiling; if they are good reflectors, then a sound will take a long time to die away and the reverberation will be long. But if the room contains many surfaces which absorb sound (such as carpets, upholstered furniture, curtains, etc.), then the reverberation time may be very short. Thus the reverberation time may be controlled by suitable choice of building materials; the plaster used for walls and ceilings is a big factor in this respect, as also are the furnishing materials, which act as absorbers of sound. The relative efficiency of these absorbing materials is assessed by comparing the effect which they have on reverberation time with that produced by an open window. A window is equivalent to a perfect absorber, for it lets all sound escape from the room.

The reciprocal of the area (in square feet) of material needed to produce the same effect on the reverberation time as one square foot of open window is called the *absorption coefficient* of the material. Thus the area of material multiplied by its absorption coefficient measures its total absorption in 'open-window units'.

When planning a concert hall it is important that the reverberation time should be the same whether the hall is empty for rehearsal, or filled with an audience. This is done by selecting a material for up-

holstering the seats so that the absorption of a seat is the same as that of the clothing of the person who will subsequently sit there.

Another factor of importance in the design of a concert hall is that the sound should be equally distributed over the whole hall. It is very common in halls built at the end of the last century to find a curved wall behind the platform. This acts as a curved mirror, and may have the beneficial effect of spreading the sound in a divergent beam over the whole hall if the curvature is small, but it may focus the sound strongly in one place if the curvature is large; domed ceilings have a similar effect. Modern practice tends to avoid curved surfaces and to use instead a number of small flat surfaces at different inclinations.

An effect which may be noticed in the vicinity of buildings is called the *Echelon Effect*. The student is most likely to have experienced this effect when walking on a paved path beside a paling fence, or when approaching a flight of steps such as is found in front of many public buildings. It is especially noticeable when wearing shoes with steel-tipped heels, so that each footstep gives a sharp percussive sound.

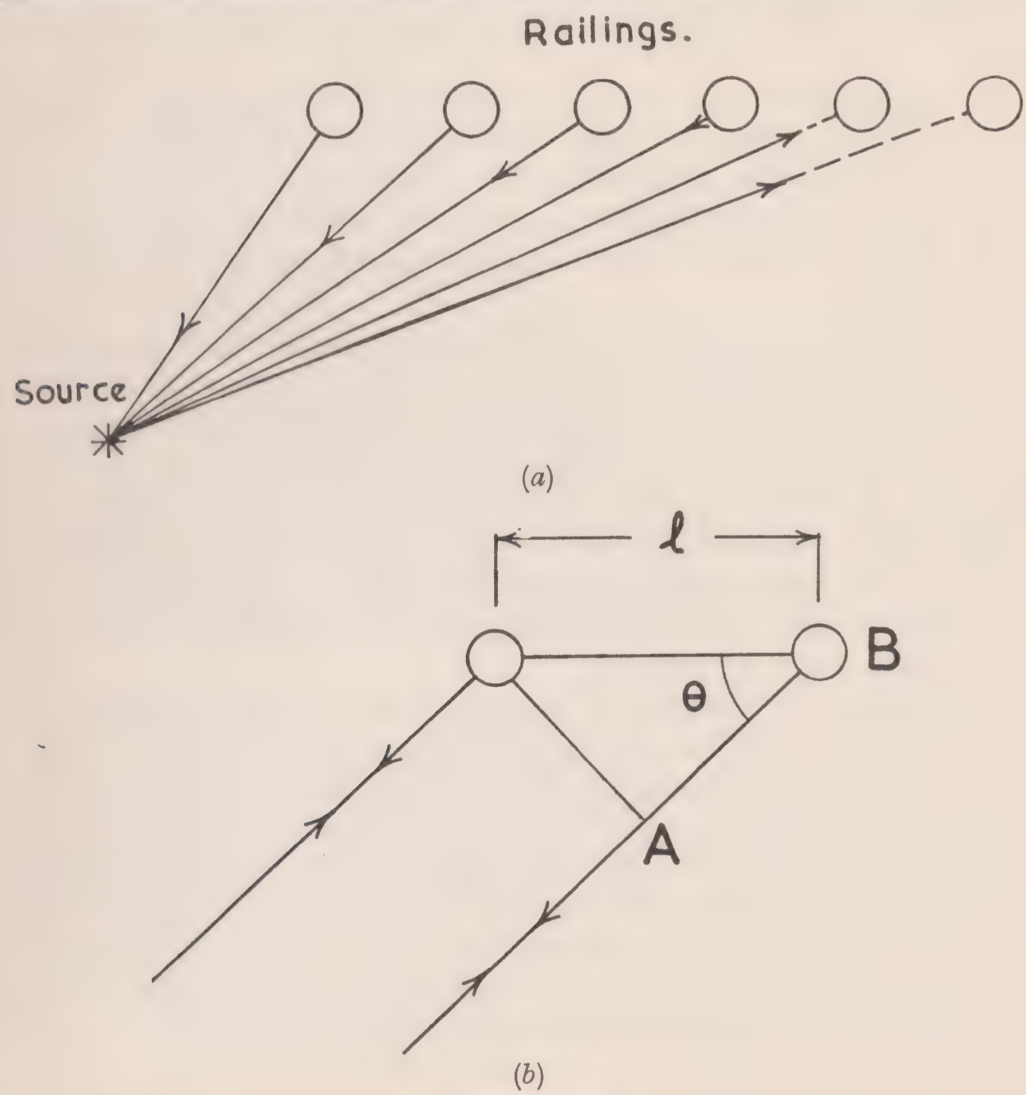


Fig. 12.34

In this case, each step seems to be followed by an echo that has a ringing quality about it.

The effect is caused by the click of the heel being reflected by each railing in turn and so one impulse is received back as a succession of pulses (see Fig. 12.34 (*a*)) which sound like a musical note. If the sound wave makes an angle θ with the plane of the fence as shown in Fig. 12.34 (*b*) and the distance between railings is l, then the path difference between two pulses is $2AB$, which is equal to $2l \cos \theta$, and thus if the velocity of sound is v, the frequency of the note is given by $v/2l \cos \theta$.

12.7 Ultrasonics

In the last few years, physicists have started to make use of waves of frequencies rather higher than those appreciated by our ears; these are called *ultrasonic* waves, and normally have a frequency between 20 and 50 kcs. They are of use because they can be focused into a narrow beam and so travel through solids and liquids with very little loss of energy due to spreading; much less than occurs with normal sound waves.

One application of this is a miniature echo-sounder which is used for finding flaws in large metal castings. An ultrasonic transmitter and a receiver placed beside it are scanned over the surface of the casting. A crack in the casting causes a reflection of the ultrasonic waves which is detected by the receiver and so the position of cracks can be plotted.

It has also been found that if the end of a rod is vibrated against a piece of metal at ultrasonic frequency, it bores a hole which may be of any shape since the 'drill' does not revolve. Also ultrasonic waves can be used to break up the oxide film normally present on the surface of aluminium, thereby exposing the metallic surface and enabling it to be soldered. Ultrasonic waves can be used to destroy bacteria in liquids, for the rapid compression and rarefaction waves burst the cells. By a similar process, air bubbles can be removed from a liquid, such as molten metal, used for castings.

The ultrasonic frequency is generated by a valve oscillator and converted into a mechanical wave motion by a *transducer*. This is either a quartz crystal using the piezo-electric effect or a bar of nickel using the *magnetostrictive* effect. When nickel is magnetised it decreases in length, thus if a rod of nickel is placed in a solenoid which is energised by an alternating current, the rod will be set into longitudinal vibration. If the frequency of the alternating current is the same as that of the natural frequency of the rod, then mechanical vibrating of fairly large amplitude at ultrasonic frequencies can be generated.

EXERCISES 12

1. Show how a curve may be used to represent the displacements of a line of air 'particles' disturbed by waves set up by a source emitting only a simple tone. Indicate on the diagram the points where (*a*) the particle velocity, (*b*) the pressure, is greatest, giving the reason in each case.

   Show how the shape of the curve would be modified if the source emitted a sound containing a fundamental tone and an overtone of twice the frequency and half the amplitude of the fundamental. When drawing the diagram assume that the phases of the two vibrations are the same at the origin. (London Univ. Inter. B.Sc.)

2. Discuss the factors which determine the velocity of transverse waves along a uniform stretched string.

   Explain why the quality of the sound emitted when a string stretched between two fixed points is plucked at the middle differs from that emitted when it is plucked at a point one-quarter of the length of the string from one end.

   Describe in detail how, using a stretched string, you would make an accurate determination of the frequency of a tuning-fork.

   (London Univ. Inter. B.Sc.)

3. Describe and explain the phenomenon of beats, and describe how you would use it to adjust a sonometer wire into unison with a tuning-fork.

   What determines (*a*) the pitch, (*b*) the quality, of a musical note? Describe ONE experiment in support of each of your answers.

   (Oxford H.S.C.)

4. Explain how you would expect a change of temperature to affect the pitch of the note emitted by (*a*) a tuning-fork, (*b*) an organ pipe.

   Two organ pipes emit notes of frequency 256 at 15° C. On raising the temperature of one of the pipes two beats per second are heard. Calculate the rise of temperature. (London Univ. Inter. B.Sc.)

5. Two organ pipes, one open, one closed, are constructed to emit the same fundamental note. Explain why the qualities of tone of the two pipes differ, and show how you would verify your explanation. How is the quality affected by the width of the pipes?

   (London Univ. Inter. B.Sc.)

6. Explain the terms *pitch, quality, loudness* and state the physical properties of sound waves on which they depend.

   A high-pitched source of sound is fixed in front of a plane reflecting surface. A sensitive flame is moved slowly along the normal joining the reflecting surface to the source and is found to 'duck' 20 times in 40 cm. Explain this and determine the frequency of the sound if the velocity of sound in air is 33,000 cm./sec. (Cambridge H.S.C.)

7. What is the nature of the difference between longitudinal and transverse waves propagated along a wire? Explain how a sine curve can be used to represent either type of motion.

   What length of wire of linear density 0·088 gm.cm.$^{-1}$ under a

tension of 40·0 kgm.wt. is required to give a note of frequency 43·2 cycles.sec.$^{-1}$ when vibrating transversely? What device is adopted in a piano to obtain such a low frequency using a wire limited in length to 150 cm.?

A stretched string of cross-sectional area a and length l is made of a material of density ρ and Young's modulus E. What is the frequency of its fundamental mode of (*a*) longitudinal, (*b*) transverse vibration? Show that the former is much greater than the latter.

(London Univ. G.C.E. Advanced level.)

8. Explain the phenomenon of beats and show that the number of beats per second due to two sources of sound is equal to the difference between the frequencies of the two sources.

   Two open organ pipes, sounding their fundamentals, give 5 beats per sec. when the temperature is 0° C. How many beats per sec. will they give when the temperature is 25° C.? Expansion of the pipes may be neglected. (London Univ. Inter. B.Sc.)

9. Describe and explain the mode of action of an electrically driven tuning-fork or vibrator. How may such a fork or vibrator be used to verify the relation between the tension and wavelength of transverse waves on a stretched string or wire?

   A fine wire 500 cm. long is fixed at both ends and is under tension, the fundamental frequency of transverse vibrations being 50 cycles per second. At what distance from its centre must a bridge be placed in order that 4 beats per second may be heard when both sections of the wire are made to vibrate transversely?

   (Northern Univ. G.C.E. Schol. level.)

10. Describe a method of producing a steady source of sound of constant frequency.

    If the source produces a high-pitched note, how would you measure its frequency? Give details of the theory and necessary calculations.

    (London Univ. G.C.E. Advanced level.)

11. A symphony played by an orchestra is received by a microphone, recorded on a gramophone disc and later reproduced through a loud-speaker. Give an account of the way in which the original sound energy is changed by each of these devices, indicating in each likely causes of deterioration in quality.

    (Northern Univ. G.C.E. Schol. level.)

12. Define the *decibel.*

    Why are two sounds of equal intensity not necessarily of equal loudness? In what circumstances will they be of equal loudness?

    Describe two instruments (one in each instance) which are used to convert (*a*) the energy of a sound wave into electrical energy, (*b*) electrical energy into sound energy. Draw a diagram for each instrument you describe. (Northern Univ. G.C.E. Advanced level.)

13. The velocity of a particle describing simple harmonic motion is 16 cm. per sec. at a distance of 8 cm. from the centre of the motion and 8 cm. per sec. at 12 cm. from the centre. Starting from the definition

of simple harmonic motion, calculate the period and amplitude of the above motion.

A particle is subjected to two simple harmonic motions, of equal amplitudes and at right angles to one another, the frequencies being in the ratio of 1 : 2. Assuming the two motions to be in phase at the start, determine, by a graphical method, the resultant motion of the particle. Describe an experiment to illustrate this motion, and indicate how the accuracy of the frequency ratio may be tested.

(Northern Univ. G.C.E. Schol. level.)

14. Describe an experiment in which two simple harmonic vibrations are combined at right angles to form Lissajous' figures. Determine graphically the resultant figure when the vibrations expressed by $y_1 = a \sin \frac{2\pi t}{3}$ and $y_2 = a \sin \frac{\pi t}{3}$ are combined at right angles, each beginning at zero displacement. (London Univ. Inter. B.Sc.)

15. What are Lissajous' figures? Draw a labelled diagram of the experimental arrangement you would adopt in order to adjust three tuning-forks so that two are in unison and the third an octave higher. Draw diagrams to illustrate the appearances of the figures as the final adjustments are gradually approached. (London Univ. Inter. B.Sc.)

16. Describe in detail the stroboscopic method of determining the frequency of vibration of a tuning-fork. *Outline* TWO other methods which are available for measuring the frequency and discuss the respective merits of all the methods you have described.

An outer ring of 12 dots and an inner ring of 8 dots, all uniformly spaced, are painted on a disc which can be rotated about its axis and which is viewed stroboscopically by means of a tuning-fork of frequency 100 vibrations per second, the disc being seen once in each vibration of the fork. Calculate the minimum rate at which the disc must be rotated so that the outer ring shall appear at rest with the normal spacing between the dots.

The disc is then speeded up until *both* rings appear to be at rest. Determine the rate of revolution when this first occurs and describe and account for any peculiarity in the appearance of the outer ring.

Mention some other application of the stroboscope principle.

(Northern Univ. G.C.E. Schol. level.)

17. Describe the determination of the frequency of a tuning-fork by a method which does not involve the use of a sonometer or resonance column.

What determines the musical interval between two notes? Describe briefly how you would demonstrate the truth of your statement.

If the frequency of the note c (doh) is 261·0 cycles sec.$^{-1}$, what is the frequency of g (soh) (*a*) on the diatonic scale and (*b*) on the scale of equal temperament? (London Univ. G.C.E. Advanced level.)

18. You are provided with two tuning-forks of nearly the same (unknown) frequency. Describe in detail how you would find the frequency of each, using *either* a stroboscopic method *or* a smoked disc

on the turntable of a gramophone, and correcting for any loading of a fork that is necessary to the experiment. (London Univ. Inter. B.Sc.)

19. Give an account of the propagation of sound in the atmosphere, discussing especially the effects of wind and of vertical temperature gradients. Describe how the velocity of sound has been measured in the open air, and mention the chief experimental difficulties in such a determination.

An organ pipe emits a fundamental note of frequency 128 sec.$^{-1}$ at 10° C.; what will the frequency of this note become if the temperature rises to 21° C.? (Oxford H.S.C.)

20. Describe, briefly, a method of measuring the velocity of sound in air.

An army sound-ranging apparatus is capable of measuring the direction of arrival of sound waves from aircraft. An aircraft flying on a straight course, at a speed of 200 m.p.h., passes 10 miles from the apparatus. Draw a graph showing how its measured direction varies with time during the flight.

(Velocity of sound = 750 m.p.h.) (Manchester Univ. Schol.)

21. Discuss the terms *intensity* and *loudness* as applied to sound.

A concert hall has a domed ceiling which is a portion of a sphere whose centre of curvature is at floor-level. Assuming the source of sound to be at this level, explain why, from the acoustical standpoint, the design is not good and would be much improved by doubling the radius of curvature of the dome without altering its maximum height above the floor.

Explain why the acoustical properties of a concert hall may vary with the size of the audience. (Northern Univ. H.S.C.)

22. (*a*) Distinguish between *echo* and *reverberation.*

(*b*) Define the reverberation time of a hall and state briefly how good reception (i) of speech, (ii) of music, depends on the magnitude of this quantity.

(*c*) Discuss and explain how the reverberation time of a hall is affected by (i) its size, (ii) its shape, (iii) the nature of its wall surfaces, and (iv) the size of an audience.

(Northern Univ. G.C.E. Advanced level.)

23. Comment on the following statements:

(*a*) The loudness of the note emitted by a tuning-fork is 10 decibels.

(*b*) The loudness of the noise in a railway carriage is 80 phons.

(*c*) Some loudspeakers do not give faithful reproduction.

(*d*) Speaking in some public halls is like speaking out-of-doors; such halls, also, are not suitable for concerts.

(Northern Univ. G.C.E. Advanced level.)

24. Discuss how the phenomena of reflection, interference and absorption of sound enter into the acoustic properties of a hall or room.

What is meant by the time of reverberation? Sabine obtained the formula

$$T = \frac{0{\cdot}05V}{A},$$

where T is the reverberation time in seconds, V the volume of the room in cu. ft. and A the absorption in sq. ft. of open window units. Comment upon the usefulness of this formula in acoustic design of auditoria. (London Univ. G.C.E. Advanced level.)

25. Explain carefully the formation of standing waves and describe the characteristics which distinguish them from progressive waves.

Describe briefly an experiment in which a standing wave system is utilised for finding the velocity of sound in a gas.

An observer approaching a flight of steps notices that each footstep is followed by an echo having a musical ring. If the frequency of the note he hears is 600 c/s, determine the width of each step; the air temperature is assumed to be 17° C.

(London Univ. G.C.E. Advanced level.)

26. State carefully the difference between progressive and stationary waves. Show that the wavelength of a progressive wave system may be determined from measurements made on a stationary wave system.

A man taps the ground in front of an iron fence with a hammer and he hears a musical sound after each tap. If the distance between successive vertical iron rods of the fence is x cm. and the man is d cm. away from the fence, find an expression for the frequency of the musical sound, and state carefully the conditions under which a sound can be heard. (Cambridge G.C.E. Schol. level.)

ANSWERS TO EXERCISES

EXERCISES 1, page 14.

1. 6.20 p.m.
2. 576 ft.
3. 11·25 miles.
4. 48 $ft.sec^{-1}$, 2·2 $ft.sec^{-2}$, 30 $ft.sec^{-1}$, 29·6 $ft.sec^{-1}$.
5. 6 sec.
6. $2\sqrt{(H/g)}$sec, 27 $gt^2/16$ ft.
7. 2·5 min, 1·5 min.
8. 14·4 min, 8 nautical miles, 37 ° S of E.
9. (i) 23·4 mph (ii) 1·6 miles.
10. (i) 113°58′ or 46°49′ (ii) 2 hr 52 min 39 sec p.m.
11. 8·7 mph, 54°12′ S of E.

EXERCISES 2, page 32.

1. tan θ, 52 sec.
2. Assuming that incline is arc sin (1/10) then:
   (*a*) weight unchanged, plumbline hangs at 5°44′,
   (*b*) weight increased by 1%, plumbline hangs at 11°21′.
3. 14·1 gm-wt, 0·742 kilowatts.
4. 28·5 $lb\text{-}wt.ton^{-1}$, 38·1 $lb\text{-}wt.ton^{-1}$.
6. 31·6 $cm.sec^{-1}$, 108°26′ to $\overrightarrow{AB}$, $5\sqrt{1000}$ $gm.cm.sec^{-1}$, 158 dynes.
7. 32·64 $ft.sec^{-1}$.
8. $kv/(1-k)$.
10. 100 ft. poundals.
11. $16{\cdot}86 \times 10^7$ dynes, 2×10^4 cal.
12. 0·0106 C deg.
13. 1 ton-wt, 3/4.
15. 36·5°.
16. 1·275 cm.
17. 791 $cm.sec^{-1}$ at 54°22′ N of W.
18. 800 hp, 7·94 mph.
19. 61·6 lb-wt, 10·08 hp.
20. 4000 hp, 46·2 sec, or 90 sec.
22. $2{\cdot}6 \times 10^5$ cal.

EXERCISES 3, page 52.

1. $W_{\max} = M/(\frac{1}{\mu}\cos\theta + \sin\theta)$, where M is mass of man and μ friction between man and ground.
4. 13 ft.
5. 30°, $g/\sqrt{3}$.
6. 0·965.
7. $\text{arc cot}\left\{\frac{1}{\mu}\left(1 - \frac{u^2}{2gh}\right)\right\}$
8. 0·35, 1·9 hp.
9. 0·77 sec, 0·69 sec.
10. 5.
11. 526 cm, 8·55 sec, 0·459 joules.
12. 132 cm, 31·9 cm.
13. 2 $cm.sec^{-1}$, 32 $cm.sec^{-1}$.
14. 34·1 $cm.sec^{-1}$ at 70°20′ to the normal.
15. 3·48 cm and 0·79 cm.

EXERCISES 4, page 107.

2. 20·2 cm above the base.
3. $2r(\sin\alpha)/3\alpha$, $4\sqrt{2}\,r/3\pi$ from centre.
4. About 0·1% error if buoyancy is ignored.
6. 19 min arc.
7. 0·210 gm, 0·724.
8. 55 min arc.
9. Front 556·6 kg-wt, back 443·4 kg-wt.
10. 50 $cm.sec^{-2}$.
11. $mr^2/2$, $\dfrac{g(m_1 - m_2)t^2}{2\left[m_1 + m_2 + \dfrac{I}{r^2}\right]}$.
12. $6\frac{2}{3}$ rev per sec.
13. (*a*) $Mr^2/2$, (*b*) $Mr^2/4$, $g/3$.
14. 800 cycles per min.
15. 2 $\mu g \cos\alpha$.
16. Gains $\frac{1}{2}$%.
17. 11·74°, 16 joules.
18. 24900 ft.lb-wt, 264 rev.
19. 38 $rad.sec^{-1}$.
20. (*a*) 23·4 $rad.\ sec^{-1}$, (*b*) 1·07 sec.
21. $Mr^2/2$, 0·017.
22. 7·2 sec, $5{\cdot}6 \times 10^7$erg.
23. Depends on assumptions made. Question implies that rotation of balls is significant, but problem is then very complex. If rotational energy is ignored and balls are smooth and perfectly elastic then:

energy of second ball after collision =

$$\tfrac{1}{2} Mu^2\left\{1 - \left[\frac{M-m}{M+m}\right]^2\right\}$$

where u = initial velocity of first ball.

Maximum transfer of energy when $M = m$.

24. 0·38 rev.sec^{-1}, 578 gm-wt.

25. $v_{\min} = \sqrt{(gr)}$,

$$h_{\max} = r + \frac{v^2}{2g} + \frac{gr^2}{2v^2} = \frac{(v^2 + v_{\min}^2)^2}{2gv^2}.$$

26. 0·68.

27. 8° from vertical.

29. 90 ft.sec^{-1}.

30. 25 ft, π sec.

31. 7·85 ft.sec^{-1}, 92·6 ft.poundal.sec^{-1}, 96·9 ft.poundal.sec^{-1}.

32. 2π sec, $(g + \sqrt{5})$ poundal, $(g - \sqrt{5})$ poundal.

33. 157 cps.

34. 0·635 sec.

35. 0·4 sec, 2 cm.

36. 0·45 sec, 7·5 × 10^5 erg.

37. 982 cm.sec^{-2}.

38. (*a*) 20 cm, (*b*) 0·1962 joules, (*c*) 0·04905 joules.

39. Loses about 56 sec per hr.

40. Loses 5·2 sec per day.

41. 2·28 sec, 0·4 ft.poundal, 42·2 lb.

EXERCISES 5, page 153.

1. 31·6 mgm-wt.

2. 2550 sec.

3. 8·9 × 10^{-2} cm.sec^{-1}.

4. 5.

5. $2\pi(mL^2/c)^{\frac{1}{2}}$,

$$2\pi\left\{mL^2\Big/\left[c + \frac{2\,G\,m\,M\,L\,d}{(L-d)^3}\right]\right\}^{\frac{1}{2}}.$$

6. 389300 km.

7. 6·70 × 10^{-8} c.g.s. units.

8. 16/81.

9. 977·4 cm.sec^{-2}.

10. (*a*) 1·52, (*b*) 0·725.

11. 5075 sec per rev.

13. 3·813 × 10^{10} cm.

14. (*a*) 1·524, (*b*) 248·4 sidereal years.

15. 0·0047% longer.

16. $\sqrt{(2/3)}$.

17. (*a*) 101·5 cm, (*b*) 4·294 × 10^5 gm.cm^2.

18. 2·76.

19. 5·7 min. arc.

20. 0·13 cm.sec^{-2}.

22. 5·376 × 10^{27} gm.

23. 2·765 km.

24. 1·107 × 10^6 cm.sec^{-1}.

25. 7·985 km.sec^{-1}, tangential; 11·29 km.sec^{-1}, any direction.

26. $$G\left[m\left\{\frac{1}{r} - \frac{1+\sqrt{(M/m)}}{a}\right\} + M\left\{\frac{1}{a-r} - \frac{1+\sqrt{(M/m)}}{a\sqrt{(M/m)}}\right\}\right].$$

(*a*) 0, (*b*) F.

EXERCISES 6, page 178.

1. 72·4 cm.

3. 34 ft.

4. $\omega^2d^2/2g$, length $(d^2 - 2gl/\omega^2)^{\frac{1}{2}}$ of horizontal tube is emptied of mercury.

5. 8 cm in benzene, 1·92 sec.

6. 19·7 ton-wt, 18 ft.

7. 0·4221, 231·2 lb.

8. (*a*) 1·40, (*b*) 1·67.

9. Volume below SG 1·00 mark 50 cm^3, area of cross-section of stem 0·59 cm^2, 8660 dynes, 65000 ergs.

10. 8·41.

11. 7·5 cm^3 in brine, 18·7 cm^3 in mixture.

12. $(2/a)(\pi m/g)^{\frac{1}{2}}$.

13. (*a*) 6·35 sec, (*b*) 6·35 sec, (*c*) 6·43 sec.

14. 6·28 m.

16. (i) 1092 lb-wt, 1·763 ft from lower edge, (ii) 3750 lb-wt, 2 ft from lower edge.

17. 33750 lb-wt, 4·22 ft from lower edge.

EXERCISES 7, page 199.

1. 3 ft, 154 ft^3.

2. 7540 m.

3. 22·5 cm, 19·24 cm.

4. 12 cm.

5. 74·6 cm of mercury.

6. 10·9 cm of mercury.

9. 3 mm of mercury per min.

10. 1·5 × 10^4 joules.

11. 2·15 km.sec^{-1}.

12. 0·465 km.sec^{-1}.

13. 8·5 × 10^9 rad.sec^{-1}.

EXERCISES 8, page 224.

1. (*a*) 2/3 in, (*b*) 3 in.
2. 2·4 kg, 16·7 cm.
3. (*a*) 1·86 mm, (*b*) 2·79 mm.
4. (i) 75 cm, (ii) 120 gm-wt in copper, 240 gm-wt in steel, (iii) $0{\cdot}655 \times 10^{-3}$ cm.
5. 0·327 cm.
6. $1{\cdot}047 \times 10^{8}$ dynes, $2{\cdot}62 \times 10^{7}$ ergs.
7. 0·375%.
8. $0{\cdot}96 \times 10^{6}$ ergs.
9. 6·24 ton-wt.
10. 2×10^{-4} cm².
11. $4d\,[1 - l/2(d^2 + l^2/4)^{\frac{1}{2}}]$.
12. $l_0\,[1 + g\,(M + \mu/2)/\lambda]$.
13. 1000·191 m.
14. 4140 kg-wt.
15. 22000 kg-wt, 77·6 joules.
16. (*a*) 704 kg-wt, (*b*) 425 kg-wt, (*c*) $4{\cdot}012 \times 10^{8}$ dyne.cm^{-2}.
17. $8{\cdot}56 \times 10^{6}$ dyne.cm, $8{\cdot}90 \times 10^{5}$ ergs.
18. 8·54 gm.cm^{-3}.
19. 9400 gm.cm².
20. 0·393 joules.

EXERCISES 9, page 270.

1. 8 cm.
2. 0·0073 cm, 0·75 cm.
3. 2·85 cm.
4. 4·43 cm.
5. 1·81 cm.
6. 0·03965 cm.
7. 2·7 cm.
8. 0·44 cm.
9. 0·1 cm.
10. $1{\cdot}024 \times 10^{6}$ dyne.cm^{-2}.
11. 68·6 dyne.cm^{-1}, 245 gm-wt.
12. 2·87 kg-wt.
13. 65·1 dyne.cm^{-1}, 1·33 cm.
14. Ambiguous question: if free bubble in air, then ST = 78·4 dyne.cm^{-1}, $\theta = 63°16'$; if bubble under liquid, then ST = 39·2 dyne.cm^{-1}, $\theta = 25°50'$.
15. $4{\cdot}54 \times 10^{5}$ dynes, force is trebled.
18. 0·8.
19. 0·01 gm.cm^{-1}.sec^{-1}.
22. $5{\cdot}5 \times 10^{7}$ sec.
23. 37·7 poise.
24. 18·9.
25. 4·55 cm.sec^{-1}.
26. 0·0251 cm.
27. 0·454 cm.sec^{-1}.
28. 22 sec.
29. (i) 0·000399 cm, (ii) 0·1749 cm.sec^{-1}.
30. $A = 3{\cdot}55$, $n = -\,0{\cdot}435$.
31. 1·84 km.

EXERCISES 10, page 309.

1. 279 m.sec^{-1}.
2. 4556·7 ft.
3. 20842 ft.
4. 255° 30′, 1800 m.
5. 7300 ft.
6. 53°.
9. 232·6 m.sec^{-1}.
10. 7396 ton-wt.in^{-2}.
11. $5{\cdot}85 \times 10^{5}$ cm.sec^{-1} away from Earth, 4861·422 A.
12. 270 cps.
13. 59 cps.
14. 1010·3 cps.
16. 4·65 cps.
17. $\lambda_A/\lambda_B = (V - v)/(V + v)$, $f_A/f_B = [(V + v)/(V - v)]^2$.
18. (*a*) 1·8 sec, (*b*) 80·6 cm.

EXERCISES 11, page 351.

1. (i) 5×10^{-6} cm, (ii) 85 cm, (iii) 72°.
2. 186·5 cps, 184·5 cps, 335·7 m.sec^{-1}.
3. 1238 cps.
4. Transverse : longitudinal :: 1 : 4.
5. 3·4321.
6. (i) 50 cm, (ii) 5th.
7. 153·8 cps, 206·4 cps.
8. 127·4 cps.
9. 6 : 5, 1·1 beat per sec.
10. 1·27 cps, 1%.
11. 8·6.
14. $l_{\text{closed}}/l_{\text{open}} = 5/4$.
15. 331·5 m.sec^{-1}.
16. 217·9 cps, 653·7 cps.
17. $l_{\text{closed}}/l_{\text{open}} = \frac{1}{2}$
18. 566·7 cps, 1·2 cm.
19. 1·395.
20. 709 cps, 10·9 cm, 82·9 cm.
21. 23·8 cm, 73·8 cm.
22. 500 cps.
23. 12·6 cm, 44·8 cm.
24. 1150 cps, 5·82 in.
27. 131·7 cps.
28. 130·1 cps.
29. – 0·0126%.

EXERCISES 12, page 401.

4. 4·5 C deg.
6. 8250 cps.
7. 27·81 cm.
8. 5·224.
9. 4·998 cm.
13. 4·055 sec, 13·06 cm.
16. 8·33 rev.sec^{-1}, 12·5 rev.sec^{-1} outer ring then has 24 dots.
17. (*a*) 391·5 cps, (*b*) 391 cps.
19. 130·5 cps.
25. 28·51 cm.
26. $2dv/x^2$ cps, where v cm.sec^{-1} is the speed of sound in air.

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